XXII. *Researches in Terrestrial Physics.*—Part II. *By* Henry Hennessy, *M.R.I.A. &c. Communicated by Major* Ludlow Beamish, *F.R.S.*

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Preliminary Remarks.

In the first part of these researches, I have endeavoured, by generalizing the hypothesis on which is usually founded the theory of the earth's figure, not only to improve that theory, but also to establish a secure basis for researches into the changes which may have taken place within and at the surface of the earth during the epochs of its geological history. Although I stated that no precise physical evidence could be adduced for the examination of the assumption that the molecules of the primitive fluid, supposed to have constituted the earth, retained their positions after solidification, it yet appears that such evidence exists, if we may be permitted to draw any conclusions relative to the physical properties of substances in the earth's interior, from the observed physical properties of substances at its surface. Professor Bischof of Bonn, has shown* that Granite contracts in volume in passing from the fluid to the solid crystalline state, from 1 to ·7481, Trachyte from 1 to ·8187, and Basalt from 1 to ·8960. The first of these rocks appears, as far as can be observed, to constitute the greater part of the earth; hence the assumption alluded to must be considered not only as superfluous, but as erroneous.

In this Part it is my object to discover relations between the interior structure of the earth and phenomena observed at its surface, and also the effects of the reaction of the fluid nucleus, described in Article 6, Part I., upon the solid crust. I divide this Part into sections, each containing a distinct investigation, the order of arrangement of these sections being determined according to their fundamental importance. The statement of the geological results capable of being deduced from these investigations is, for greater clearness, reserved for the end. Such of these results as chiefly depend on the validity of the reasonings used in Section III. are presented with some diffidence, owing to the imperfect experimental knowledge we possess respecting the subjects discussed in that section. The diminution of the earth's mean radius by refrigeration is neglected all through, except where the contrary is specially mentioned.

I. THE PRESSURES OF THE SHELL AND NUCLEUS AT THEIR SURFACE OF CONTACT.

1. In this investigation the earth is supposed to consist of a nucleus of fluid matter inclosed in a solid shell. The inner and outer surfaces of the shell are sup-

* Leonhard und Bronn's Neues Jahrbuch, 1841, p. 565. See also a paper by the same writer in the N. Jahrbuch for 1843, p. 1. These results seem also to be confirmed by others subsequently found by M. Deville.

posed to be spheroidal, oblate, and nearly spherical. Both shell and nucleus are supposed to be formed of strata increasing in density, according to unknown laws, as the radii of these strata decrease.

If the earth were solid from its surface to its centre, all the phenomena of its rotation could be determined by the integration of three differential equations relative to its motion about its centre of gravity, and of three more relative to the motions of its principal axes referred to three rectangular axes fixed in space. The constants contained in these equations would depend on the attractions of exterior bodies, on the impressed forces, and on the arrangement of the particles composing the earth. It is evident that similar equations would suffice for the case of an empty shell constituted in the manner above mentioned, after the substitution of the constants depending on the magnitude and internal constitution of the shell for those depending on the magnitude and internal constitution of the entirely solid spheroid. By adopting this method, no new analytical transformations are required in discovering the phenomena of the shell's independent rotation, for we can thus avail ourselves of the researches already made by geometers relative to the rotation of the entirely solid spheroid. I have therefore thought it sufficient for our present purposes to merely present the following conclusions to which I have arrived, by using the method above indicated.

1st. If the original impressed forces were the same in direction and intensity for the shell as for the entirely solid spheroid, the angular velocity of the former about its instantaneous axis would be greater than the angular velocity of the latter about its instantaneous axis.

2nd. The influence of exterior disturbing forces would be insensible relative to the rotation of the shell about its centre, provided that its strata possess nearly similar laws of arrangement with those of the solid spheroid.

3rd. The motions in space of the axes of the shell due to the action of exterior disturbing bodies, would be affected to a greater degree than similar motions of the axes of the solid spheroid.

If no other forces acted on the fluid nucleus besides the attractions of its own particles, the attraction of the shell, and centrifugal force, its surface would be an ellipsoid of revolution, and it would rotate steadily about the axis of its greatest moment of inertia. If the action of exterior disturbing bodies be added to these forces, tidal oscillations in the surface of the nucleus would tend to be produced; but it is evident, that unless the disturbing forces were very great, the position in space of the axis of rotation of the mass would be much less affected than if it existed in the solid state. Let, in addition to the forces already enumerated as acting on the fluid nucleus, certain molecular forces be conceived to act on its particles, by which the whole mass might in general acquire a tendency to expand or contract, and also to change its form. This class of forces may be conceived to be resolved into two pressures acting at the inner surface of the shell, one of them being supposed to be constant for every point, and the other variable. If, moreover, the constitution of the shell and nucleus should be favourable to a great amount of friction at their surface of contact, it is at least possible to conceive that if the sum of the pressures existing at the same surface be sufficiently great, the whole mass might rotate nearly as if solid from its surface to its centre. As it will hereafter appear that such conditions must almost necessarily exist, it is superfluous for our present object to further examine the general problem of the rotation of the shell and nucleus. I shall therefore proceed to consider the pressures which might take place at the surface of contact of the shell and nucleus.

2. Abstracting the action of exterior disturbing bodies, the pressure of the nucleus at the inner surface of the shell will result from the attractions of all the particles of the shell and nucleus, centrifugal force, and molecular actions. Unless the last mentioned class of forces should have a tendency to disturb the position of the axes of rotation of the shell and nucleus, it is plain that these axes may be considered as coincident.

Of the two resultant pressures mentioned in the preceding article, it is evident that the variable pressure may be conceived to result from the difference in form of the free surface of the nucleus and the shell's inner surface. If these surfaces be very nearly spheroids of revolution, described about the shell's axis of rotation, it follows that the greatest pressure will be at the equator or poles of the shell, according as the tendency of the nucleus may be to become more or less oblate. In either case it is evident that every point on certain lines situated between the equator and poles will not be subjected to any pressure from this cause, or in other words, the sum of all the pressures at the points in question will be equal to the constant pressure. The general expression for the pressure at any point of the shell may therefore be made to contain the coordinates of the lines in question as constant quantities. As these lines in the case considered are evidently circles parallel to the equator of the shell, we may call them the parallels of mean pressure.

Let a straight and indefinitely narrow canal of fluid be conceived to reach from the centre to the surface of the nucleus. Let the origin of the coordinates of any point in it be fixed at the centre of gravity of the mass, and let the plane of x, y be perpendicular to the axis of rotation. Let r represent the radius drawn from the origin to a molecule in the canal, θ the angle made by this radius with the axis of rotation, and ω the angle formed by the projection of the radius on the plane of x, y with the axis of y; then

$$z=r\cos\theta,\ y=r\sin\theta\cos\omega,\ x=r\sin\theta\sin\omega.$$

The pressure p at the point in the canal having these coordinates will be expressed by the equation

$$dp=\varrho[\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz+\alpha^2(xdx+ydy)],$$

in which, as in art. 2, Part I., X, Y, Z represent the components of the attractions parallel to the three rectangular axes of x, y and z, α the angular velocity of rotation of the nucleus, and ϱ the density at the point in the canal. This equation may be written as follows, after the substitution of the above values, and it being remembered that θ and ω are constant for the canal,

$$dp=\varrho(d\mathrm{V}+\alpha^2 \sin^2\theta . rdr),$$

V being a function of r, θ and ω. On integration this gives

$$\mathrm{P}-\mathrm{P}'=\int_0^r \varrho d\mathrm{V}+\alpha^2 \sin^2\theta \int_0^r \varrho rdr,$$

P standing for the pressure on an unit of surface of the stratum of the nucleus having the radius r, and P'_0 the pressure at the centre. With the angular velocity α_1, all other things remaining as before,

$$\mathrm{P}''-\mathrm{P}'''=\int_0^r \varrho d\mathrm{V}+\alpha_1^2 \sin^2\theta \int_0^r \varrho rdr.$$

But when θ becomes θ_1, its value at the parallel of mean pressure,

$$\mathrm{P}'=\Pi-\int_0^{r'} \varrho d\mathrm{V}-\alpha^2 \sin^2\theta_1 \int_0^{r'} \varrho rdr,$$

$$\mathrm{P}'''=\Pi-\int_0^{r'} \varrho d\mathrm{V}-\alpha_1^2 \sin^2\theta_1 \int_0^{r'} \varrho rdr,$$

Π representing the constant pressure on the stratum in question, and r' the value of r at the parallel of mean pressure. If α_1 be that angular velocity which would cause the surface of the nucleus to coincide with that of the shell $\mathrm{P}''=\Pi$, we shall obtain by combining the foregoing expressions and neglecting the difference between r and r',

$$\mathrm{P}=\Pi+(\alpha^2-\alpha_1^2)(\sin^2\theta-\sin^2\theta_1)\int_0^r \varrho rdr. \quad \ldots \ldots \quad (1.)$$

For the pressure P_1 at the shell's inner surface, we shall have

$$\mathrm{P}_1=\Pi_1+(\alpha^2-\alpha_1^2)(\sin^2\theta-\sin^2\theta_1)\int_0^{r_1} \varrho rdr, \quad \ldots \ldots \quad (2.)$$

r_1 and Π_1 representing respectively the radius and pressure at the surface stratum of the nucleus. If Π_1 be not negative, this expression will be always positive with respect to some portion of the shell's inner surface; for when $\alpha > \alpha_1$ the greatest pressure is at the equator, and $\theta > \theta_1$; when $\alpha < \alpha_1$, the greatest pressure is at the poles, and consequently $\theta < \theta_1$.

3. The determination of θ_1 may be thus easily effected. Let the equations of the generating ellipses of the surfaces of the nucleus and shell be respectively

$$\mathrm{A}_1^2x^2+\mathrm{B}_1^2z^2=\mathrm{A}_1^2\mathrm{B}_1^2,\ \mathrm{A}_2^2x'^2+\mathrm{B}_2^2z'^2=\mathrm{A}_2^2\mathrm{B}_2^2,$$

A_1, B_1 and A_2, B representing respectively the less and greater axes of these surfaces. The volumes contained within these surfaces being equal, we have $\mathrm{A}_1\mathrm{B}_1^2=\mathrm{A}_2\mathrm{B}_2^2$, and hence at the parallels of mean pressure, where $x=x'$ and $z=z'$,

$$z=\pm \mathrm{A}_1\mathrm{A}_2\sqrt{\frac{\mathrm{B}_1^2-\mathrm{B}_2^2}{\mathrm{B}_1^2\mathrm{A}_2^2-\mathrm{B}_2^2\mathrm{A}_1^2}}.$$

This expression shows that the parallels of mean pressure are symmetrically situated at each side of the equator.

If for z its value be substituted, and the condition of equality of volumes be remembered, we shall obtain

$$\cos\theta_1 = \pm\frac{A_2}{\sqrt{A_1^2+A_1A_2+A_2^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

r_1 being supposed to differ very little from A_1.

If the changes in the oblateness of the nucleus be small, $\cos\theta_1$ will be nearly constant, and the parallels of mean pressure will oscillate but slightly about their mean position. Hence if $A_2=A_1(1+\varepsilon)$, ε will be a very small quantity depending on the difference of the ellipticities of the two surfaces, but by TAYLOR's theorem

$$F(A_2)=F(A)+\varepsilon F'(A)+\frac{\varepsilon^2}{1.2}F''(A_1)+\ldots.$$

Hence if the surface of the shell should change with every change of form of the nucleus, ε will be infinitely small, and consequently

$$\cos\theta_1=\pm\frac{1}{\sqrt{3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

If the surface of the shell should not change its form for every change in form of the nucleus, ε might yet be so small as to be negligible compared with other quantities entering into our analytical expressions, and hence in such a case the above expression for θ_1 will be approximately true.

As θ_1 is the complement of the latitude at the parallel of mean pressure, we may conclude from the preceding investigation, that, *at the parallel the square of the sine of the latitude of which is one-third, the pressure of the fluid is always the same as if the surface of the nucleus were one of equal pressure for the shell.*

Hence if we represent by f the centrifugal force at the equator of the shell's outer surface, where the radius is supposed equal to unity, and by f_1 its value corresponding to a_1, equation (1.) becomes

$$P=\Pi-(f-f_1)\left(\cos^2\theta-\frac{1}{3}\right)\int_0^a \varrho a da, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

where a_1 represents the radius of a sphere equal in volume to the spheroid included within the stratum of the nucleus with the radius r, and which consequently differs but little from r.

The simple expressions for the pressure on any stratum of the nucleus and on the shell's inner surface obtained in this section, will be found useful in the course of these researches, and particularly in the succeeding section.

II. THE VARIATION OF GRAVITY AT THE EARTH'S SURFACE.

4. It is in general evident, that if the laws of arrangement of the molecules composing the shell and nucleus be different, the variation of gravity at the surface of

the former will not be the same as if in solidifying all the particles retained the same positions which they had when constituting the entirely fluid mass. All investigations heretofore made of the variation of gravity at the earth's surface being grounded on this very untenable hypothesis, it seems desirable that a more general solution of the question should be obtained, in which such a supposition would not be involved.

The forces acting on a particle at the shell's outer surface which we have here to examine, are—1st, the shell's attraction; 2nd, the attraction of the nucleus; 3rd, centrifugal force. If the laws of arrangement of the matter composing the shell be continuous, the first of these forces will be equal to the difference between the attraction of the entirely solid spheroid and that of the spheroid produced by the complete solidification of the nucleus, the particles in both spheroids arranging themselves according to the influence of the forces acting on them.

5. *Attraction of the Shell.*—In a spheroid differing but little from a sphere, and composed of homogeneous strata varying in form and density, the expression upon which its attraction on an exterior point depends, is thus written*,—

$$V=\frac{4\pi}{r'}\int_0^{a'}\varrho_2 a^2 da+\frac{4\pi\beta}{r'}\int_0^{a'}\varrho_2 d\left(a^3W_0+\frac{a^4}{3r}W_1+\&c.\right), \quad \ldots \ldots \quad (6.)$$

the radius of each stratum being of the form $a(1+\beta w)$, a being the radius of the sphere equal in volume to the mass included within the surface of that stratum, and $w=W_0+W_1+W_2+\&c.$, W_0, W_1, &c. being functions satisfying the equation of LAPLACE's coefficients, ϱ_2 the density of any stratum, a' the value of a at the surface, and r' the radius of the attracted point.

If the above expression be supposed to refer to the spheroid included within the shell's outer surface, it is evident that a corresponding expression may be obtained for the spheroid included within the shell's inner surface by merely changing β into β', and a' into a_1; β' being a constant depending on the ellipticity of that surface, and a_1 the value of a corresponding to it, we shall have therefore

$$V_1=\frac{4\pi}{r'}\int_0^{a_1}\varrho_2 a^2 da+\frac{4\pi\beta'}{r'}\int_0^{a_1}\varrho_2 d\left(aW_0+\frac{a^4}{3r}W_1+\text{etc.}\right). \quad \ldots \ldots \quad (7.)$$

If (6.) and (7.) be differentiated with respect to r, and the first then subtracted from the second, we shall obtain for G_1, the attraction of the shell, the expression

$$G_1=\frac{4\pi}{r^2}\int_{a_1}^{a'}\varrho_2 a^2 da+\frac{4\pi}{r'^2}\left\{\beta\int_0^{a'}\varrho_2 d\left(a^3W_0+\frac{2a^4}{3r}W_1+\frac{3a^5}{5r^2}W_2+\ldots\right)\right.$$
$$\left.-\beta'\int_0^{a_1}\varrho_2 d\left(a^3W_0+\frac{2a^4}{3r}W_1+\frac{3a^5}{5r^2}W_2+\ldots\right)\right\} \quad \ldots \ldots \quad (8.)$$

Let us conceive the surfaces of both spheroids to be covered with a homogeneous fluid to a small depth compared with their radii. The bounding surfaces of these fluid strata will depend on the attractions of the spheroids, and also on the attractions of the fluid particles. If the density of each of these strata be nearly the same as that

* See PONTÉCOULANT, Théorie Analytique, &c., Livre V. No. 23.

of the solid stratum in immediate contact with it, its superposition on the spheroid will evidently not much increase the mass of the entire attracting body. The equation of equilibrium at the surface of the fluid on the greater spheroid is

$$\int_0^{a'_1}\frac{dp}{\varrho_2}=\frac{4\pi}{3a'_1}(1-\beta w)\int_0^{a'_1}\varrho_2 da^3+\frac{4\beta\pi}{a'_1}\int_0^{a'_1}\varrho_2 d\Big[a^3 W_0+\frac{a^4}{3r}W_1+\frac{a^5}{5r^2}W_2+\text{etc.}\Big]$$
$$+\frac{1}{3}fa_1'^2-\frac{1}{2}fa'^2\Big(\cos^2\theta-\frac{1}{3}\Big),$$

a'_1 being the value of a' at the fluid's surface, and p the pressure on the stratum. On developing and then comparing similar terms, we obtain

$$\int_0^{a'_1}\frac{dp}{\varrho_2}-\frac{4\pi}{3a}\int_0^{a'_1}\varrho_2 d.a^3+\frac{4\pi\beta W_0}{3a'_1}\int_0^{a'_1}\varrho_2 da^3-\frac{4\beta\pi}{a'_1}\int_0^{a'_1}\varrho_2 da^3 W_0-\frac{1}{3}fa_1'^2=0.$$

The strata of the shell being bounded by spheroids of revolution and the origin of the coordinates being at the centre of gravity,

$$Wi=0,$$

i being different from 2. From the equation for W_0, it can be immediately inferred that W_0 may receive an arbitrary value. For W_2 we obtain

$$\frac{4\beta\pi}{5a_1'^3}\int_0^{a'_1}\varrho_2 d.a^5 W^2-\frac{4\pi\beta W_2}{3a'_1}\int_0^{a'_1}\varrho_2 da^3+\frac{1}{2}fa_1^2\Big(\cos^2\theta-\frac{1}{3}\Big)=0,$$

whence

$$\beta W_2=-\beta h\Big(\cos^2\theta-\frac{1}{3}\Big), \quad \quad (9.)$$

in which

$$\beta h=mf(a'_1), \quad \quad (10.)$$

m standing for the ratio of centrifugal force to gravity at the equator, and f being a functional symbol. If we make $W_0=-\frac{1}{3}h$, and substitute in the expression for r these values, we would have

$$r'=a'_1(1-\beta h\cos^2\theta),$$

and therefore βh must represent the ellipticity of the fluid stratum.

Similarly for the surface of the fluid stratum on the smaller spheroid, where $\overline{a}_1$ represents what $\overline{a}_1$ becomes for the fluid

$$r_1=\overline{a}_1(1-\beta' h\cos^2\theta),\quad \beta' h=mf(\overline{a}_1). \quad \quad (11.)$$

Besides these general expressions for βW_0, &c., $\beta' W_0$, &c., it is possible to find others in a particular case, which expressions will subsequently be found useful. The particular case referred to, is that in which the shell's strata are supposed to have all the same ellipticity. On this hypothesis W_0, W_1, &c. are not functions of a, and consequently (8.) becomes

$$G=\frac{4\pi}{r^2}\int_{a_1}^{a'}\varrho a^2 da+\frac{4\pi\beta}{r^2}\Big[3W_0\int_0^{a'}\varrho a^2 da+\frac{8}{3}\frac{W_1}{r}\int_0^{a'}\varrho a^3 da+\frac{3W_2}{r^2}\int_0^{a'}\varrho a^4 da+\dots\Big]$$
$$-\frac{4\pi\beta'}{{}^2}\Big[3W_0\int_0^{a_1}\varrho a^2 da+\frac{8}{3}\frac{W_1}{r}\int^{a_1}\varrho a^3 da+\frac{3W_2}{r^2}\int_0^{a_1}\varrho . a^4 da+\dots\Big].$$

From the values attributed to a' and a_1, W_0 will disappear, and from the origin chosen for the coordinates W_1 will also disappear*.

Let M represent the mass of the greater spheroid, I its moment of inertia with respect to its axis of rotation, M_1 and I_1 being the corresponding quantities relative to the smaller spheroid, then

$$G=\frac{M-M_1}{r^2}+\frac{\beta}{r^4}\left\{\frac{q}{2}IW_2+c_3W_3+\ldots\right\}-\frac{\beta'}{r^4}\left\{\frac{q}{2}I_1W_2+c'_3W_3+\ldots\right\},$$

c_3 and c'_3 being constant coefficients.

The artifice already employed may be used here in order to find expressions for βW_2, $\beta' W_2$, &c. In this case the equation of equilibrium of the fluid at the surface of the greater spheroid is

$$C=\frac{M'}{r'_1}\left\{1+\frac{\beta}{r'^2_1}\left[\frac{3}{2}\frac{I'}{M'}W_2+\ldots\right]\right\}-ra'^2_1f\left(\cos^2\theta-\frac{1}{3}\right),$$

M' and I' standing for the mass and moment of inertia respectively of the whole mass composed of the fluid stratum and solid spheroid. This becomes, on developing r, and making $C=\frac{M'}{a'_1}+\frac{a'^2_1}{3}f$,

$$\frac{\beta w}{a}=\beta\left[\frac{3}{2}\frac{W_2}{a^3}\frac{I}{M}+\ldots.\right]-\frac{a'^2f}{2M'}\left(\cos^2\theta-\frac{1}{3}\right).$$

But also

$$w=W_2+W_3+W_4+\ldots.+W_i.$$

The truth of these two simultaneous expressions requires that

$$W_2=\frac{3}{2}\frac{W_2}{a^2}\frac{I'}{M'}-\frac{a^3f}{2M\beta}\left(\cos^2\theta-\frac{1}{3}\right).$$

But $I=\frac{2}{5}\frac{Ma^2}{\sigma}$, σ being a number depending on the internal constitution of the spheroid, and $\frac{fa'^2}{M'}=m$ nearly; hence neglecting very small quantities, or making $a'_1=1$,

$$\beta W_2=-\frac{5\sigma m}{2(5\sigma-3)}\left(\cos^2\theta-\frac{1}{3}\right). \qquad (12.)$$

And in a similar manner we may obtain

$$\beta' W_2=-\frac{5a_1^3\sigma_1 m}{2(5\sigma_1-3)\mu_1}\left(\cos^2\theta-\frac{1}{3}\right), \qquad (13.)$$

μ_1 being used for brevity to represent $\frac{M_1}{M}$, and σ_1 being a number analogous to σ. Now if r be developed in (8.), and all terms of the order β^2 be neglected, it will become, remembering that $r'=a'(1+\beta y)$, y being a function of the polar coordinates of the point,

$$G_1=M\left\{1-\mu-\beta\left[2(1-\mu)y-\frac{q}{5}\left(\frac{1}{\sigma}-\frac{\mu\beta' a_1^2}{\beta\sigma_1}\right)W_2\right]\right.$$

$$=M\left\{1-\mu+m\left[2(1-\mu)e-\frac{qm}{2(5\sigma-3)}\left(1-a_1^5\,\frac{5\sigma-3}{5\sigma_1-3}\right)\right]\left(\cos^2\theta-\frac{1}{3}\right),\right. \qquad (14.)$$

* See the work already cited, No. 21.

it being remembered that the surface of the earth is a spheroid of revolution,

$$\beta y = -e\left(\cos^2\theta - \frac{1}{3}\right),$$

e representing the ellipticity. The expression for the attraction of a homogeneous shell with surfaces of equal ellipticity, is evidently

$$G_1 = M\left\{1 - a_1^3 + \frac{1}{2}\left[5(1-a_1^3)e - \frac{qm}{2}(1-a_1^4)\right]\right)\cos^2\theta - \frac{1}{3}\Big)\Big\}. \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

6. *Attraction of the Nucleus.*—The forces acting on any stratum of the nucleus, by changing the arrangement of its particles, must in general influence their action on exterior bodies; hence to obtain the attraction of the nucleus, it is necessary to take all such forces into consideration. These forces are in the present case, the attractions and pressures of masses of fluid within and without the stratum, the attraction of the shell, and centrifugal force. Hence for any stratum we have the following equation of equilibrium,

$$\begin{aligned}\int\frac{dp}{p} = \frac{4\pi}{3r}\int_0^a \varrho da^3 + \frac{4\beta\pi}{r}\int_0^a \varrho d\left(a^3 Y_0 + \frac{a^4}{3r}Y_1 + \frac{a^5}{5r^2}Y_2 + \text{etc.}\right)\\ +2\pi\int_a^{a_1}\varrho d.a^2 + 4\beta_1\pi\int_0^{a_1}\varrho d\left(a^2 Y_0 + \frac{ar}{3}Y_1 + \frac{r^2}{5}Y_2 + \text{etc.}\right)\\ +2\pi\int_{a_1}^{a'}\varrho_2 d.a^2 + 4\beta\pi\int_{a_1}^{a'}\varrho_2 d\left(a^2 W_0 + arW_1 + \frac{r^2}{5}W_0 + \text{etc.}\right)\\ +\frac{1}{3}fr^2 - \frac{1}{2}fr^2\left(\cos^2\theta - \frac{1}{3}\right),\end{aligned}$$

where p represents the pressure at any point in the stratum; Y_0, Y_1, &c. are functions of the coordinates satisfying the equation of LAPLACE's coefficients, and also possessing the property of forming the terms of the series into which the radius of the point may be developed.

But as the nucleus is inclosed in a rigid shell, its surface is constrained to take a form different from that which it would have if the shell were removed. The integral at the left side of the above equation is consequently variable. The conditions of equilibrium require that it should be constant, and these conditions may be fulfilled by separating the variable from the constant terms in that quantity and transposing the former terms to the right side, the quantities Y_0, Y_1, &c. being supposed to undergo any variations which may be required by the new conditions.

By article 2,

$$p = \Pi - (f - f_1)\left(\cos^2\theta - \frac{1}{3}\right)\int_0^a \varrho a da,$$

Π being such a pressure that

$$p_1 = \Pi - (f - f_1)\sin^2\theta_1\int_0^{a_1}\varrho a da,$$

θ_1 being that value of θ which makes $p = \Pi$, and p_1 a function of a_1, and consequently a constant for the nucleus.

On developing $r=a(1+\beta y)$, and substituting the value of p, we obtain

$$\int_0^a \frac{dp_1}{\varrho}=\frac{4\pi}{3a}(1-\beta y)\int_0^a \varrho.da^3+\frac{4\beta\pi}{a}\left[\int_0^a \varrho d.a^3\text{Y}+\frac{1}{3a}\int_0^a \varrho.da^4\text{Y}_1+\frac{1}{5a^2}\int_0^a \varrho.da^5\text{Y}_a+\text{etc.}\right]$$

$$+2\pi\int_a^{a_1} \varrho da^2+4\beta\pi\left[\int_a^{a_1} \varrho.da^2\text{Y}_0+\frac{a}{3}\int_a^{a_1} \varrho da\text{Y}_1+\frac{a^2}{5}\int_a^{a_1} \varrho d\text{Y}_2+\text{etc.}\right]$$

$$+2\pi\int_{a_1}^{a'} \varrho da^2+4\beta\pi\left[\int_{a_1}^{a'} \varrho da^2\text{W}_0+\frac{a}{3}\int_{a_1}^{a'} \varrho da\text{W}_1+\frac{a^2}{5}\int_{a_1}^{a'} \varrho d\text{W}_2+\text{etc.}\right]$$

$$-\frac{1}{2}(f-f_1)a^2\sin^2\theta-\frac{1}{2}fa^2\left(\cos^2\theta-\frac{1}{3}\right)+\frac{1}{3}fa^2.$$

But

$$y=\text{Y}_0+\text{Y}_1+\text{Y}_2+\ldots.\text{Y}_i.$$

Hence if this value be substituted in the above equation, and if after substitution, similar functions of θ and ω, the coordinates of the point in the nucleus be equated to zero, we shall find

$$\int\frac{dp_1}{\varrho}-\frac{1}{3}fa^2-\frac{1}{2}(f_1-f)a^2-\frac{4\pi}{3a}(1-\beta\text{Y}_0)f\varrho d.a^3-\frac{4\ \pi}{a}\int_0^a \varrho da^3\text{Y}_0$$

$$-2\pi\left(\int_0^{a_1} \varrho da^2+\int_{a_1}^{a_1} \varrho_2 da^2\right)-4\pi\left(\beta\int_0^a \varrho.d.a^2\text{Y}_0+\beta\int_{a_1}^{a'} \varrho_2 da^2\text{W}_0\right)=0.$$

But (art. 5) as $\text{W}_0=0$, and $\int\frac{dp_1}{\varrho}$ is constant, this expression shows that Y_0 may receive an arbitrary value. $\text{Y}_1=0$ by the property of the centre of gravity, which is the origin of the coordinates. When $i=2$,

$$\frac{4\beta_1\pi a^2}{5}\int_a^{a_1} \varrho.d\text{Y}_2+\frac{4\beta_1\pi a^2}{5}\int_{a_1}^{a'} \varrho_2 d\text{W}_2-\frac{4\beta_1\pi}{3a}\text{Y}_2\int_0^a \varrho.da^3+\frac{4\beta\pi}{5a^3}\int_0^a \varrho.da^5\text{Y}_2$$

$$+\frac{1}{2}(f-f_1)a^2\cos^2\theta-\frac{1}{2}fa^2\left(\cos^2\theta-\frac{1}{3}\right)=0.$$

But we have already found

$$\beta\text{W}_2=-mf(a')\left(\cos^2\theta-\frac{1}{3}\right),$$

hence the above becomes

$$\frac{4\beta_1\pi a^2}{5}\int_a^{a_1} \varrho d\text{Y}_2-\frac{4\beta\pi\text{Y}_2}{3a}\int_0^a \varrho d.a^3+\frac{4\beta\pi}{5a^3}\int_0^a \varrho da^5\text{Y}_2+\frac{1}{2}\left(f_1\cos^2\theta-\frac{1}{3}f\right)a^2$$

$$-\frac{a^2 m}{a}\Psi(a_1 a')\left(\cos^2\theta-\frac{1}{3}\right),$$

where $\Psi(a_1, a')$ is a function depending on the attraction of the shell, or in other words, depending on the arrangement of the shell's strata.

When i is any number greater than 2, we have

$$\frac{4\pi a^i}{2i+1}\left[\beta_1\int_a^{a_1} \varrho\frac{d\text{Y}_i}{a^{i-2}}+\beta\int_{a_1}^{a'} \varrho_2\frac{d\text{W}i}{a^{i-2}}\right]-\frac{4\pi\beta}{3a}\text{Y}_i\int_0^a \varrho.da^3+\frac{4\pi}{(2i+1)a^{i+1}}\int_0^a \varrho da^{i+3}\text{Y}_i=0.$$

But as $\text{W}_i=0$, and as the surface of the nucleus is a spheroid of revolution, $\text{Y}_i=0$. At the surface of the nucleus

$$\frac{4\beta\pi}{5a_1^3}\int_0^{a_1}\varrho d.a^5 Y_2-\frac{4\beta\pi Y_2}{3a_1}\int_0^{a_1}\varrho d.a^3+\frac{1}{2}a_1^2 f_1\Big(\cos^2\theta-\frac{1}{3}\Big)-a_1^2 m\Psi(a_1,a')\Big(\cos^2\theta-\frac{1}{3}\Big), \quad . \quad (a.)$$

because

$$f_1\Big(\cos^2\theta-\frac{1}{3}\frac{f}{f_1}\Big)=f_1\Big(\cos^2\theta-\frac{1}{3}\,\frac{1}{1+f'}\Big)=f_1\Big[\cos^2\theta-\frac{1}{3}(1-f')\Big],$$

where f' is a very small quantity, and where consequently $f_1 f'$ is a negligible product.

Solving (*a*.) with respect to β_1 Y_2, we obtain

$$\beta_1 Y_2=-\beta h\Big(\cos^2\theta-\frac{1}{3}\Big), \ (16.), \qquad \beta_1 h=f_1\Phi(a_1)+a_1^2 m\Psi(a_1,a'). \quad . \quad . \quad . \quad . \quad (17.)$$

If we make $Y_0=\frac{1}{3}h$ and substitute the value of y in r, we shall have

$$r=a_1(1-\beta h\cos^2\theta).$$

Hence $\beta h=e_1$, e_1 being the ellipticity of the surface of the nucleus, and consequently of the shell's inner surface.

The general form of the function upon which depends the attraction of the nucleus on an exterior point, may be found by substituting these values in (7.), having first changed ϱ_2 into ϱ and W_0, W_1, &c. into Y_0, Y_1, &c.; hence

$$V_3=\frac{M_1}{r'}\Big\{1-\frac{e_1}{r}-\frac{e_1}{r^2}\Big(\cos^2\theta-\frac{1}{3}\Big)a_1^2\Big\}+\frac{a_1^5}{r^3}\Big[\frac{1}{2}f_1-m\Psi(a_1,a')\Big]\Big(\cos^2\theta-\frac{1}{3}\Big), \quad . \quad . \quad (18.)$$

assuming that the mass of the nucleus is equal to the mass of the solid spheroid included within its surface, having the same law of density as the shell. The truth of this assumption will appear more manifest further on.

7. If, in a similar manner, we substitute the values of βW_2, &c., $\beta' W_2$, &c. in (8.), then develope r in both (8.) and (18.), after the differentiation of (18.) we shall have the sum of the attractions of the nucleus and shell on a point at the shell's outer surface, by adding the resulting expressions. If to this sum we add the term produced by the centrifugal force, we shall have for G, gravity at any point of the surface having θ for the complement of its latitude, the expression

$$G=G'(1+\lambda\cos^2\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.)$$

where

$$\lambda=m\Big[\frac{5}{2}-3\big(f(a)-\mu_1 a_1^2 f(a_1)\big)\Big]+\frac{3}{2}\Big[m_1-m(1+\Psi(a',a_1)\Big]a_1^5+2e-3e_1\mu a_1^2, \quad . \quad . \quad (20.)$$

all terms of the second order being as usual neglected, m_1 standing for the value of m corresponding to f_1, and G' for gravity at the equator. When the mass is supposed to be entirely fluid $a_1=a'=1$, $m_1=m$, $\Psi(a')=0$, $\mu=1$, and consequently $\lambda=\frac{5}{2}m-e$, as should be expected. The foregoing is the most general expression for the variation of gravity at the earth's surface yet obtained; gravity at any point being expressed as a function of the latitude of that point, of the radii and ellipticities of the shell's inner and outer surfaces, and of functions depending on the constitution of the shell and nucleus. Its value is not merely speculative, for it will be found to assist in explaining certain apparent anomalies detected by observation in the variation of gravity at the earth's surface, as well as in pointing out the limits assigned by observation to the thickness of the solid crust.

III. THE LAWS OF DENSITY OF THE SHELL AND NUCLEUS.

8. The density ϱ of a stratum of the mass when entirely fluid has been shown in Part I.* to depend on the pressure to which it may be subjected, and to the molecular properties of the fluid. The density of a stratum of the nucleus must evidently depend on the same circumstances, and hence we may assume its expression to have the same general form as for the entirely fluid mass, but yet containing variable indeterminate coefficients.

If the solidification of the nucleus proceeded entirely from its centre to its surface no shell could at any time exist, and as it will appear that it could not simultaneously solidify both from its centre towards its surface, and from its surface towards its centre, we can here consider only the latter case. When the solidification of the nucleus proceeds in this manner, its superficial stratum in contact with the inner surface of the shell will be the first to assume the solid state. As solidification must proceed very slowly from the slowness of the refrigeration of the entire mass, it is evident that the thickness of the stratum may be considered indefinitely small compared with the radius of the nucleus. Let its density in the fluid state be ϱ_1, in the solid state ϱ_2, and let $\varrho_1=k\varrho_2$, k being a function depending on the contraction of the fluid when solidifying.

If, in conformity with the preceding remarks, we assume

$$\varrho=\frac{c_1 \sin an_1}{a},$$

$c_{\prime}$ and $n_{\prime}$ varying with $a_{\prime}$ the mean radius of the nucleus, we shall have to determine four quantities in order to arrive at a knowledge of the laws of density of the shell and nucleus. It appears, however, that the number of known conditions which these quantities must satisfy will not suffice for their complete determination, although it is possible to conceive how a new condition could be experimentally found by which the required number would be made up. For if the physical properties of the matter composing the earth's interior resemble those of the matter at its surface, the form of k could be found with some degree of approximation, by a series of experiments on the contraction of fused matter at different densities resulting from differences of pressure. The conditions which can be at present determined are easily found thus: If the law of density of the shell be continuous, which must result from its mode of formation, and if the variations of a' and a_1 from refrigeration be neglected, we shall

* Articles 6 and 7.

have

$$4\pi\int_0^{a'}\varrho_2 a^2 da = 4\pi\Big(\int_{a_1}^{a'}\varrho_2 a^2 da + \int_0^{a_1}\varrho a^2 da\Big),$$

and consequently

$$\int_0^{a_1}\varrho_2 a^2 da = \int_0^{a_1}\varrho a^2 da, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.)$$

$$C_1 = \frac{n_1^2\int_0^{a_1}\varrho_2 a^2 da}{\sin a_1 n_1 - a_1 n_1 \cos a_1 n_1};$$

and from the condition $\varrho_1 = k\varrho_2$,

$$k = \frac{\frac{n_1^2}{a_1\varrho_2}\int_0^{a_1}\varrho_2 a^2 da}{1 - a_1 n_1 \cot a_1 n_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.)$$

9. Let now an indefinitely thin canal of fluid be conceived to reach from the centre in a straight line to the surface of the nucleus, and consequently to pass through the surface stratum having the thickness da_1. Conceive this stratum to consist of an infinite number of elementary parallelopipeds, having their bases resting against the shell's inner surface. As the oscillations of the shell's surface are small, the simultaneous contraction of all these parallelopipeds can take place only in a direction perpendicular to their bases, and consequently da_1 will become $k_1 da_1$, k_1 being the cubical contraction of a mass of the fluid composing the stratum. The increase in volume of the shell, by the addition of the new stratum, will be less than the decrease in volume of the nucleus by its abstraction from its mass; and hence the nucleus must tend to expand in order to fill the empty space which would otherwise exist. This would evidently result as a necessary consequence of the cause of the variation in density of the strata from its centre to its surface.

The radius of the nucleus after the solidification of its superficial stratum being $a_1 - kda_1$, and it being manifest that the entire mass of the nucleus before the solidification took place is equal to the mass of the solid stratum with the thickness kda_1, and the mass of the new nucleus,

$$\int_0^{a_1}\varrho a^2 da = \int_0^{a_1 - kda}\varrho' a^2 da + \int_{a_1 - kda_1}^{a_1}\varrho_2 a^2 da,$$

or

$$\int_{a_1 - kda_1}^{a}(\varrho_2 - \varrho)a^2 da - \int_0^{a_1 - kda_1}(\varrho - \varrho')a^2 da = 0,$$

where ϱ' is the density of the stratum with the radius a, after the solidification of the superficial stratum. As ϱ_2 and ϱ are constant in the interval $k_1 da$, we have

$$(\varrho_2 - \varrho_1)k_1 a_1^2 da_1 = \int_0^{a_1 - k_1 da_1}(\varrho - \varrho')a^2 da. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23.)$$

But ϱ being a function of a, and ϱ' being what this function becomes after the expansion of the stratum to which it refers, we may write

$$\varrho = f(a), \quad \varrho' = f(a + c_1 k_1 da),$$

c_1 being indeterminate, hence

$$\varrho-\varrho'=-c_1k_1\frac{d\varrho}{da}da-\frac{c_1^2k_1^2}{1.2}\,\frac{d^2\varrho}{da^2}da^2-\dots$$

consequently (23.) becomes after substitution and differentiation with respect to a_1,

$$\frac{d\varrho_2}{da}-c'\frac{d\varrho_1}{da}+\frac{2(1-k_1)}{a}\varrho_2=0, \quad \dots \dots \quad (24.)$$

neglecting terms of the second order, and making $c_1-1=-c'$.

If we assume $\varrho_2=\frac{C_2\sin a_1n_2}{a_1}$, C_2 and n_2 being constants, and make $q_1=1-a_1n_1\cot a_1n_1$, $q_3=1-a_1n_2\cot a_1n_2$ (24.) becomes

$$1-k_1-\frac{1}{2}q_3+\frac{1}{2}c'-\frac{\varrho_1}{\varrho_2}q_1=0\,;$$

but as k_1 and k are evidently identical, we shall obtain after reduction

$$k=\frac{q_3-2}{c'q_1-2}, \quad \dots \dots \quad (25.)$$

whence

$$c'=q_2-\frac{2[1-(k)]}{(k)q}, \quad \dots \dots \quad (26.)$$

in which

$$q=1-n\cot n,\ \ q_2=1-n_2\cot n_2,$$

n being the value of n_1 at the surface of the entirely fluid mass, and (k) the value of k determined by experiment at the surface under atmospheric pressure.

But from (22.) we have

$$k=\left(\frac{n_1}{n_2}\right)^2\frac{q_3}{q_1}; \quad \dots \dots \quad (27.)$$

hence for the determination of n_1 we obtain

$$n_1^2\left(c'-\frac{2}{q_1}\right)=\left(1-\frac{2}{q_3}\right)n_2^2. \quad \dots \dots \quad (28.)$$

In obtaining these results, some assumptions had to be made which in the absence of experimental data can be considered only provisional; but it will be satisfactory to perceive that the results themselves seem to be in accordance with what we would expect from known physical laws. When $a_1=0$, $q_1=0$, $q_3=0$ and $k=1$, its greatest value, k consequently increases as the thickness of the shell increases, or as the density of the surface stratum of the nucleus increases. But the cubical contraction is equal to the cubical dilatation of the mass if reduced again to the fluid state under the same pressure, and by a recognized physical principle the dilatation of a body is equal to its resistance to compression, or inversely proportional to its compressibility. But the compressibility of a stratum of the nucleus decreases as its density increases, hence the above result appears to be in conformity with an observed physical law.

Equation (28.) may be thus written,

$$m'^2\left(c'-\frac{2}{q_1}\right)=a_1^2n_2^2\left(1-\frac{2}{q_3}\right), \quad \ldots \ldots \ldots \quad (29.)$$

making $n_1=\frac{m'}{a_1}$; from which it is evident that m' decreases with a_1, and consequently that the less the volume of the nucleus, the more does it approach the state of homogeneity.

10. If any grounds existed for believing that experiment could never even approximately disclose the laws of density and contraction of the matter composing the earth's interior, we should not be justified in attaching much importance to any hypothesis, however plausible, which may be formed relative to the forms of these laws. Such grounds would partially exist as long as doubts could be formed relative to the great or small specific density of the matter of the earth's interior. The variation in density of the spheroidal strata composing the earth, could be due to an accumulation near its centre of substances possessing a high specific density. From what we know of the matter composing the earth, I believe that almost all substances having a density much greater than that of the outer portion of its crust are metals, bodies which from their chemical nature do not probably exist to any great extent within the globe. Recent experiments tend to show that many metals are but compound radicals which exist only under certain conditions, and it is probable that the constituents of any one of these radicals have densities less than that of the metal itself. KANE, LAURENT and GERHARDT are of opinion, for instance, that the known metals do not exist in their oxides, but are eliminated when the oxides are decomposed. The metals alluded to are also simple bodies in the sense generally used, namely, that of being the lowest terms as yet known of compound bodies. Recent chemical researches seem to show that such simple bodies existed in the earth's primitive state, even in less quantity than we can at present discover them*. The chemical laws in virtue of which this would be true, would under certain conditions act as well in the interior as at the earth's surface, if we are entitled at all to admit their generality, and hence we must conclude that the specific density of the substances in the earth's interior must be subordinate to the effects of mechanical and physical causes in producing the observed variation in the density of its strata.

IV. THE FORMS OF THE STRATA OF THE SHELL.

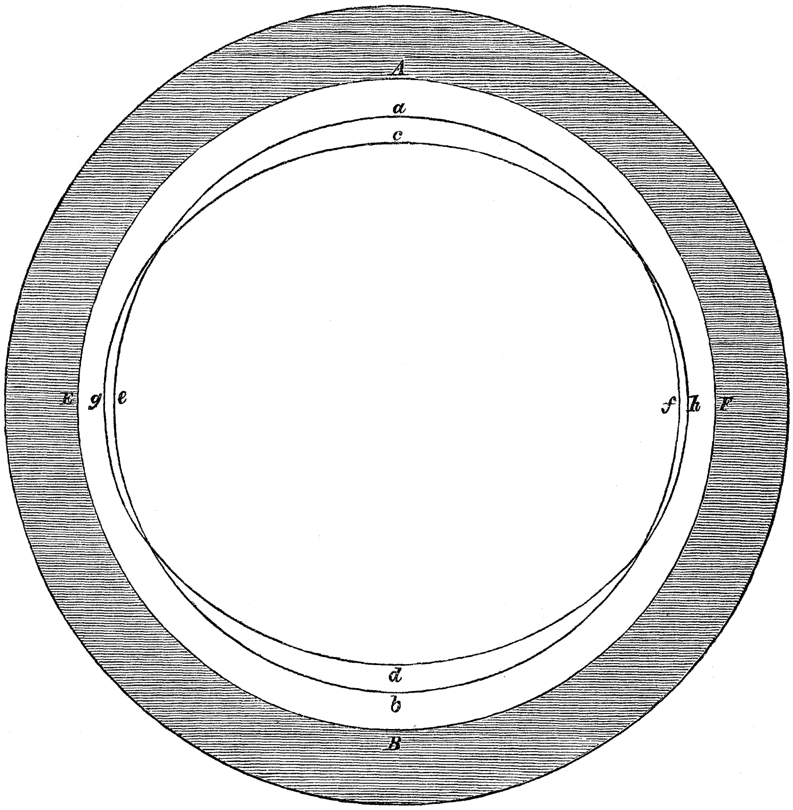
11. As every stratum of the shell was originally the surface stratum of the fluid nucleus, its form must depend on that of the shell's inner surface, and of the isothermal surfaces passing through its poles and equator. At first I shall abstract the influence of the isothermal surfaces in order to simplify the problem of the determination of the form of the stratum.

As in art. (1.), the shell is here supposed to be rigid, and to be perfectly filled

* See BISCHOF, Lehrbuch der Chem. und Phys. Geologie, Bd. I. s. 584, and Bd. II. s. 6.

with the fluid matter, but here we cannot consider the transition from solidity to fluidity as abrupt. Many physical analogies lead us to believe that, on the contrary, this transition must be gradual, or that a stratum of matter not completely fluid must exist between the highly fluid matter of the nucleus and the solid matter of the shell. The transition from this imperfectly fluid stratum to the perfectly fluid matter within it, will be also gradual, but it is possible to conceive that a portion of it in immediate contact with the shell may merely possess that kind of fluidity which in solids would be considered extreme softness. This matter will cohere to the shell with a considerable force, compared to the cohesion of the fluid particles; and this fact, combined with its viscidity, must serve to remove any doubt as to its not being subjected to the same hydrostatical laws as the perfect fluid. If now a surface be conceived to exist which may be called the effective surface of separation of the perfect from the imperfect fluid, it is evident that its form will depend on the pressures which the former tends to exert.

Let AEBF in the accompanying figure represent the profile of the inner surface of the shell, *aebf* of the similar and concentric surface of the perfect fluid. In



this case the pressure exerted by the nucleus would be the same for all points of the shell, and the thickness of the stratum of imperfect fluid would be everywhere the same. If, however, the pressure of the perfect fluid were not constant, its surface would tend to assume a different form, in order to re-establish equilibrium; it might for example, tend to assume the form of which *cgdh* is the profile. The thickness of the stratum of imperfect fluid would then be greatest at A, and least at E.

On solidifying, the new stratum thus added to the shell would have the same proportional thickness, and its interior surface would in this particular case be more elliptical than its exterior surface. If, as seems extremely probable, the imperfectly fluid stratum be thin, and if it strongly adhere to the shell, it will exercise no sensible pressure on the perfect fluid, and must consequently take the form impressed on it without any sensible resistance.

12. It can be easily shown that in general the pressure of the perfect fluid will not be constant. Let a spheroidal mass of fluid be conceived, consisting of nearly similar spheroidal strata, these strata increasing in density as their radii decrease. The ellipticities of the bounding surfaces of any stratum will depend on the constitution and thickness of the strata outside it. If possible let the mass outside the stratum be removed, without altering the law of density of the remaining fluid: it is manifest that equilibrium will be obtained only where the surface of the stratum at any point becomes perpendicular to the resultant of all the forces acting on that point. With the same law of density as the entire mass, and urged by the same forces, the ellipticity of its surface must be the same as that of the surface of the primitive mass. If, however, the law of density changed in such a way as to render the remaining fluid more homogeneous, the ellipticity of its surface would be greater than that of the primitive fluid spheroid*. Hence we may conclude—(1.) that if the angular velocity and law of density of the nucleus remained unchanged after the formation of the first stratum of the shell, the outer and inner surfaces of that stratum would be similar, and its attraction on the interior mass would consequently, by a well-known theorem, be evanescent. The next formed stratum would thus also have similar surfaces, and so on with every stratum, until the mass should have completely solidified. In this case, therefore, the surfaces of all the strata would have the same ellipticity as the outer surface. (2.) If the forces acting on the nucleus after the solidification of the first stratum of the primitive fluid were such as to give the surface of the nucleus a tendency to become less oblate, the ellipticity of the interior surface of the imperfectly fluid stratum would be less than that of its exterior surface, and the resulting attraction of the shell after the solidification would manifestly tend to increase in the same direction the effect of the forces previously in action. (3.) If, on the contrary, the resultant of all these forces were such as to increase the oblateness of the perfect fluid, it is evident, from the preceding considerations, that the inner surface of the shell so produced would always be more elliptical than its outer surface.

These conclusions will be necessarily modified when the influence of the isothermal surfaces in the interior of the spheroid is considered. It seems that no complete and general solution of the problem of finding the forms of these surfaces has been yet

* See POISSON, Mécanique, ii. p. 546, 2nd edition.

achieved, but an approximate solution, founded on an hypothesis to which no weighty objections can be urged, has been given by Mr. HOPKINS*. This solution will suffice for my present purpose, particularly when I only refer to the general result of his analysis. The result in question is, that the oblateness of the isothermal surfaces increases with their distance from the earth's surface. This result, combined with the first of the three conclusions in the foregoing paragraph, seems to show that if the earth solidified from its surface to its centre without changing its law of density, the ellipticity of the shell's inner surface would be greater than that of its outer surface. The truth of the second conclusion would be considerably weakened, while that of the third would be strengthened to the same amount.

13. It will immediately appear that this last is the conclusion which must be definitely adopted, if we admit that the matter composing the nucleus becomes denser in assuming the solid state.

Using the notation of art. 6, and neglecting the effect of isothermal surfaces, the ellipticity of the surface of the perfect fluid may be generally expressed thus,

$$e_1 = m_1 \Phi(a_1) \pm m\Psi(a_1, a'),$$

the double sign being placed before $m\Psi(a_1 a')$ to show that, according as the ellipticity of the shell's inner surface is greater or less than that of its outer surface, this term should be added or subtracted, as must appear from the theory of the attraction of spheroids. If, when e_1 is supposed to increase, the small term alluded to be neglected, the truth of any conclusion as to the rapid increase of e_1 deduced from an examination of the remaining term will be only rendered still more manifest.

Let at any period of the shell's existence the surface of the nucleus be supposed to coincide with the shell's inner surface, then $m_1 = m$, and consequently, from the expressions given in art. 6, we may deduce

$$e_1 = \frac{e_1 m a_1^3 \int_0^{a'} \varrho_2 a^2 da'}{2\left\{e_1 \int_0^{a_1} \varrho_2 a^2 da - \frac{1}{5a^2} \int \varrho . da^5 e_1\right\}} = \frac{m a_1^3 \int_0^{a'} \varrho_2 a^2 da}{2\left(1 - \frac{3}{5\sigma_2}\right) \int_0^{a_1} \varrho_2 a^2 da - \frac{2e_1}{5a_1^2} \int_0^{a_1} \varrho a_1^5 da} = \frac{m a_1^3}{2\mu\left(1 - \frac{3}{5\sigma_2}\right)},$$

σ_2 being a number depending on the law of density of the nucleus, and analogous to σ of article 5. But as the centrifugal force at the surface of the spheroid is proportional to the square of its angular velocity, and as the angular velocity is inversely proportional to the moment of inertia of the mass, we shall have

$$m = m' \left(\frac{\text{I}'}{\text{I}}\right)^2,$$

I' representing the moment of inertia of the mass in its state of entire fluidity, I the moment of inertia corresponding to m, and m' the value of m corresponding to I'. From the general expression for the moment of inertia of a spheroid, we can write

$$\frac{\text{I}'}{\text{I}} = \frac{\frac{a'^2}{\sigma_1'}}{\frac{a'^2}{\sigma} - \mu a_1^2 \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)}$$

(σ_2' being the value of σ_2 when $a_1 = a'$).

* Philosophical Transactions, 1842, p. 45.

Hence, making $a'=1$,

$$e_1=\frac{m'}{2}\left(\frac{\sigma}{\sigma'_2}\right)^2\left\{\frac{a_1^3}{\mu}+a_1^5\sigma\frac{(\sigma_2-\sigma_1)}{\sigma_1\sigma_2}\right\}^2\left(1-\frac{3}{5\sigma_1}\right)^{-1}=\frac{m'}{2}\left(\frac{\sigma}{\sigma'_2}\right)^2\left[\nu^2+2\nu a_1^5\sigma\frac{(\sigma_2-\sigma_1)}{\sigma_1\sigma_2}\right]\left(1-\frac{3}{5\sigma_1}\right)^{-1},$$

where ν^2 represents a number which always differs but little from 1. The coefficient of a_1^5 being small, it is evident that the numerator of this expression cannot vary rapidly with a_1.

From article 12, it is evident that e_1 must always either increase or decrease as a_1 increases; in order to know how it varies, it will therefore be only necessary to examine its value for any two values of a_1.

Let $a_1=1$, then

$$e_1=\frac{m'}{2}\left(\frac{\sigma}{\sigma'_2}\right)^2\left[1+2\frac{(\sigma'_2-\sigma)}{\sigma_2}\right]\left(1-\frac{3}{5\sigma}\right)^{-1}=\frac{9}{10}\,\frac{m}{(\sigma)^2}\text{K},$$

(σ) being determined by observation, and K having the value assigned to it in art. 8, Part I. It will appear that the greatest value of K which can be assigned will make e a little less than $\frac{1}{300}$.

When $a_1=0$,

$$e'=\frac{5}{2}m\left(\frac{\sigma}{\sigma'_2}\right)^2\frac{\nu_1^2\sigma'_1}{5\sigma'_1-3},$$

σ'_1 and ν_1 being nearly equal to unity; and σ'_2 and σ not much differing, e' will not much differ from m, and consequently it will be greater than e. This remark can be verified by actual calculation, when we shall have obtained the numerical values of the quantities contained in the preceding expression.

V. THE PRINCIPAL MOMENTS OF INERTIA OF THE EARTH.

14. The dependence of all the phenomena of the earth's rotation upon its principal moments of inertia, and the resulting connexion which thus subsists between these phenomena and the internal constitution of the earth, renders a complete investigation of their values of some importance for the objects to be fulfilled by the present memoir.

Let the moment of inertia with respect to the axis of rotation be C, and A and B those for the other two axes. Let C_1, &c. represent the moments of inertia of the solidified spheroid included within the shell's outer surface, C_2, &c. the moments of inertia of the spheroid included within the shell's inner surface, and C_3, &c. the moments of inertia of the nucleus, then

$$C=C_1-C_2+C_3,\quad B=B_1-B_2+B_3,\quad A=A_1-A_2+A_3,$$

and representing an element of the earth's mass by dm,

$$C=\int(x^2+y^2)dm,\quad B=\int(x^2+z^2)dm,\quad A=\int(y^2+z^2)dm,$$

x, y, z being the rectangular coordinates of the element dm. In finding C_1, &c., C_2, &c., C_3, &c., different values for dm must in general be introduced in the foregoing expressions, and then the integrals may be taken between the limits proper for each particular case.

Let x, y and z be transformed into polar coordinates as in article 2, and the sums of their squares taken two and two be put in such a form as to satisfy the equation of LAPLACE'S coefficients. We shall then have, putting μ for cos θ,

$$C_1=\int_{-1}^{+1}\int_0^{2\pi}\int_0^r \varrho_2 r^4\left\{\frac{2}{3}+\left(\frac{1}{3}-\mu^2\right)\right\}drd\omega d\mu,$$

$$B_1=\int_{-1}^{+1}\int_0^{2\pi}\int_0^r \varrho_2 r^4\left\{\frac{2}{3}+\left[\frac{1}{3}-(1-\mu^2)\sin^2\omega\right]\right\}drd\omega d\mu,$$

$$A_1=\int_{-1}^{+1}\int_0^{2\pi}\int_0^r \varrho_2 r^4\left\{\frac{2}{3}+\left[\frac{1}{3}-(1-\mu^2)\cos^2\omega\right]\right\}drd\omega d\mu.$$

But

$$\int_0^r \varrho_2 r^4 dr=\frac{1}{5}\int_0^{a'}\varrho_2\frac{d(r^5)}{da}da=\frac{1}{5}\int_0^{a'}\varrho_2\frac{d}{da}[a^5(1+\beta\omega)]da,$$

and

$$W=W_0+W_1+W_2+\dots+W_i.$$

From article 5 $W_i=0$, except when $i=2$ or 1; and in the latter case it is made to disappear when a' is the radius of a sphere equal in volume to the spheroid; we have, therefore, on making

$$\int_0^{a'}\varrho_2\frac{d(a^5)}{da}da=\chi(a'),$$

and

$$\int_0^r \varrho r^4 dr=\frac{1}{5}\chi(a')+\int_0^{a'}\varrho_2 d\frac{(a^5\beta W_2)da}{da}=\frac{1}{5}\chi(a)-5m\left[f(a')\int_0^a \varrho_2 a^2 da-\frac{1}{2}\int_0^{a'}\varrho_2 a^2 da\right]\left(\frac{1}{3}-\mu^2\right)$$

by article 5. Consequently

$$\left.\begin{aligned}
C_1&=\frac{4}{3}\left[\frac{2\pi}{5}\chi(a')-\frac{Mm}{3}\left(f(a')-\frac{1}{2}\right)\right],\\
C_2&=\frac{4}{3}\left[\frac{2\pi}{5}\chi(a_1)-\frac{Mm}{3}\left(f(a_1)\mu_1-\frac{1}{2}a_1^5\right)\right],\\
C_3&=\frac{4}{3}\left[\frac{2\pi}{5}\chi_1(a_1)-\frac{Mm}{3}\left(e_1a_1^2\mu_1-\frac{1}{2}m_1a_1^5\right)\pm ma_1^5\Upsilon(a_1,a')\right],
\end{aligned}\right\}\quad\dots\dots\quad (30.)$$

where

$$\chi_1(a_1)=\int_0^{a_1}\varrho\frac{d(a^5)da}{da}.$$

Hence

$$C=\frac{4}{3}\left\{S+\frac{M\Upsilon}{3}\right\},\quad\dots\dots\quad (31.)$$

making for brevity

$$S=\frac{2\pi}{5}[\chi(a')-\chi(a_1)+\chi_1(a_1),\quad\dots\dots\quad (32.)$$

$$\Upsilon=e_1\mu_1a_1^2-\frac{1}{2}m_2a_1^5-m\left[\frac{1}{2}-f(a')+\mu_1a_1^2f(a_1)-\left(\frac{1}{2}+\Upsilon(a_1,a')a_1^5\right]\quad\dots\quad (33.)$$

By following a similar process we obtain

$$A=\frac{4}{3}\left\{S-\frac{M\Psi}{6}\right\},\ B=\frac{4}{3}\left\{S-\frac{M\Psi}{6}\right\},$$

and consequently

$$\frac{2C-A-B}{C}=\frac{\Psi}{S}M,$$

neglecting quantities of the second order.

On substituting for ϱ_2 its value (art. 9.),

$$\chi(a')-\chi(a_1)=\frac{5C_2}{n_2^4}\{[2n_2^2+(n_2^2-6)q_2]\sin n_2-[2a_1^2n_2^2+(a_1^2n_2^2-6)q^3]\sin a_1n_2\};$$

and similarly, on substituting for ϱ its value in $\chi_1(a_1)$,

$$\chi_1(a_1)=\frac{5a_1\varrho_2k}{n_1^4}[2a_1^2n_1^2+(a_1^2n_1^2-6)q_1];$$

but by articles 8. and 9,

$$C_2=\frac{(\varrho_2)}{\sin n_2},\quad \varrho_2=(\varrho_2)\frac{\sin a_1n_2}{\sin n_2},\quad M=4\pi(\varrho_2)q_2n_2^2,$$

(ϱ_2) being the value of ϱ_2 at the shell's outer surface. Hence, finally,

$$S=\frac{M}{2q_2}\left\{\left[2+\left(1-\frac{6}{n_2^2}\right)q_2\right]-\left[2a_1^2+\left(a_1^2-\frac{6}{n_2^2}\right)q_3\right]\frac{\sin a_1n_2}{\sin n_2}\right.$$
$$\left.+\left[2a_1^2+\left(a_1^2-\frac{6}{n_1^2}\right)q_1\right]\frac{kn_2^2\sin a_1n_2}{n_1^2\sin n_2}\right\}.$$

But we have, by article 9,

$$k=\frac{n_1^2}{n_2^2}\frac{q_3}{q_1};$$

and on comparing equations (20.) and (23.),

$$\Psi=\frac{2e+m-\lambda}{3}; \quad \ldots \quad (34.)$$

hence if we make

$$\frac{2C-A-B}{C}=p',$$

we shall obtain

$$\left\{2a_1^2+\left(a_1^2-\frac{6}{n_2^2}\right)q_3-\frac{q_3}{q_1}\left[2a_1^2+\left(a_1^2-\frac{6}{n_1^2}\right)q_1\right]\right\}\sin a_1n_2=Q,$$

or

$$\left\{2a_1^2\left(1-\frac{q_3}{q_1}\right)-6\left(\frac{1}{n_2^2}-\frac{a_1^2}{n'^2}\right)q_3\right\}\sin a_1n_2=Q, \quad \ldots \quad (35.)$$

making for brevity

$$Q=\left[2+\left(1-\frac{6}{n_2^2}\right)q_2-\frac{2}{3}\cdot\frac{2e+m-\lambda}{p'}q_2\right]\sin n_2. \quad \ldots \quad (36.)$$

But the value of p' is given from the phenomena of the precession of the equinoxes and the nutation of the earth's axis, and that of λ may be obtained from experiments made with the pendulum on the earth's surface, or from the inequalities in the moon's motion in longitude and latitude depending upon the form and internal constitution of the earth. Thus an expression for finding a_1 has been obtained in which e_1 is absent, the elimination having been effected between the expressions depending on the variation of gravity at the earth's surface and the precessional phenomena.

As $C-A=C-B=\frac{2}{3}M\Psi$, it follows that the difference of the principal moments of inertia of the earth is proportional to Ψ. But from the general expression for Ψ and from Section IV., it is evident that Ψ increases as a_1 diminishes, hence we may in general conclude that *the difference between the greatest and least moment of inertia of the earth increases as the thickness of the shell increases.* This conclusion being independent of any knowledge of the absolute laws of density of the earth's interior, deserves particular attention.

VI. ON THE EXISTENCE OF A SOLID NUCLEUS WITHIN THE EARTH.

15. In the preceding investigations the earth has been supposed to solidify solely from its surface towards its centre, but it is possible to conceive how from the enormous pressure on its central strata solidification could also proceed from the centre towards the surface. If the influence of pressure in promoting solidification were sufficiently great, the earth might have solidified entirely from its centre towards its surface, according to the theory proposed by POISSON. It becomes important therefore to examine how far we are justified in adopting the theory of solidification first mentioned.

If solidification took place from the centre towards the surface alone, we should believe the earth to be now entirely solid. The forms of the solid strata, composing the spheroid in this case, would not be in general the same as if the original fluid mass had solidified from its surface towards its centre. Before the solidification of any part of the mass, it would consist, in virtue of hydrostatical laws, of a series of spheroidal strata of equal pressure. Refrigeration at the centre would proceed at an almost insensible rate from the impediments to convection adduced in article 6, Part I., and from the necessarily slow conducting power of the fluid; and the theory of solidification examined requiring the predominance of pressure over refrigeration as an agent in solidifying the mass, it must follow that the forms of the isothermal surfaces would have little influence on those of the solidified strata. For greater simplicity, I abstract, in the first instance, the effects of refrigeration in contracting the solid nucleus and the surrounding fluid.

The first small nucleus solidified would evidently be bounded by the surface of equal pressure due to its radius. No change can take place therefore in the direction of the resultant of the forces acting on any molecule of the stratum in contact with the solid nucleus, and hence on solidifying it will retain its form. The next stratum must become solid, similarly, without changing its form, and so on towards the surface. Hence the ellipticities of the strata of the solid spheroid would be the same as when it existed in a fluid state.

If the effect of refrigeration in contracting the matter of the entire spheroid be now considered, it is evident that it will tend to lessen the mean radius of the entire mass, and consequently to increase its angular velocity of rotation. The ellipticities of the strata of fluid surrounding the nucleus would be greater after every new addition to its mass than they were before that addition took place, and therefore, after the complete solidification of the entire mass, its strata would increase in ellipticity from its centre to its surface more rapidly than the strata of the original fluid mass increased in ellipticity. In this case, the variation of gravity at the surface of the earth in going from its equator to its poles, should, by the theory of the attraction of spheroids, be greater than for the same mass in a fluid state with the same ellipticity at surface. Observation however shows that, on the contrary, the variation of gravity at the earth's surface is less rapid than for the primitively fluid spheroid. The earth could not therefore have solidified in this manner required by this theory.

16. If, by the action of refrigeration and compression, solidification proceeded simultaneously from the surface towards the centre, and from the centre towards the surface, it will not be doubted but that the former process, when once commenced, must proceed far more rapidly than the latter. The temperature at the centre would be nearly constant, compared with the temperature of the surface of the fluid, from the obstacles opposed to convection and the laws of propagation of heat by conduction. Solidification at the centre will therefore be due chiefly to pressure; but if contraction accompany the change of state of the fluid matter in becoming solid, it appears from Section III. that the pressure on every stratum of the fluid will decrease with the radius of the shell's inner surface. The pressure on the solid nucleus will thus be continually diminishing, while its temperature will remain almost unchanged. The solid nucleus would therefore, instead of acquiring increased magnitude, tend to return to its original fluid state. This will evidently be true at every period of the existence of the nucleus, and hence we must conclude that this mode of solidification is incompatible with our original assumptions.

VII. THE DIRECTIONS OF THE FISSURES IN THE SHELL WHICH MIGHT BE PRODUCED BY THE ACTION OF THE PRESSURES CONSIDERED IN SECTION I.

17. If the pressure of the nucleus against any part of the shell be sufficiently intense to produce a fissure, the direction of that fissure will depend on those of the tensions resulting from the fluid pressure and on the physical structure of the shell. In the actual case of the earth it is probable that the highly crystalline structure, which we have reason to believe is characteristic of the shell, would give to some portions of it a tendency to fracture in particular directions. We have at present no precise physical reasons for thinking that this tendency should follow any general law, so that it must now be considered as only a source of irregular deviations in the directions of the fissures from those which they would have if the shell possessed an uniform cohesive strength.

Abstracting these irregular causes, an examination of the question of the directions of the fissures in a spheroidal and nearly spherical shell, composed of strata each possessing in itself an uniform cohesive strength, but differing in this respect from every other, would assist in pointing out what should be the predominant directions of the fissures, if the greater effectiveness of general over particular causes be allowed.

The problem which is here to be examined is therefore simply to find the directions of the fissures in a thin, solid and nearly spherical shell, resulting from the pressures of a mass of fluid inclosed in it, the cohesion of the shell being supposed to vary, according to some continuous law, along a vertical line from its inner to its outer surface.

If the constant pressure referred to in Section I. alone acted on the shell, the tensions at any point in one of the shell's indefinitely thin strata would be equal to each other and situated in the tangent plane at that point. The position of the maximum resultant tension would therefore be indeterminate, and consequently there would be no tendency to form a fissure in any particular direction rather than in any other. If the variable pressure be superimposed on the constant pressure, the direction of the maximum resultant tension would be determinate, and consequently also the line in which a fissure would commence. At first, if the intensity of the pressures should gradually increase, a portion of the shell at each side of the equator bounded by parallels would be subjected to tensions sufficiently great to produce fissures, while beyond these parallels the shell would remain unfractured. The portion where the tensions would be sufficiently great to produce fissures, would thus constitute a disturbed district with nearly fixed boundaries. At a point where there is a tendency towards the formation of a fissure, the direction of the maximum resultant tension will be in the direction of the tangent to the meridian; the greatest tendency to form a fissure will therefore be parallel to the equator. From the nature of the variable pressure the maximum tensions must be equal, at equal distances from the equator; hence such a fissure, when once commenced, would tend to be propagated along a parallel of latitude until the force of the tensions become sufficiently lessened by the separation of the extended portion of the shell. Similar fissures would be formed simultaneously and symmetrically at each side of the equator. As long as the tensions in the directions of the tangents to the meridians continued sufficiently great, such fissures would be formed; but, as already mentioned, their formation would tend to annul these tensions, and a new system would result, having a tendency, as may be readily deduced from Mr. HOPKINS's investigations*, to produce fissures perpendicular to those previously formed. If the maximum intensity of the variable pressure be not inconsiderable compared to the constant pressure, it will follow, if the pressures continue to act with sufficient energy, that all the shell's fissures will be either parallel or perpendicular to the equator.

If, on the contrary, the constant pressure were far greater than the variable pres-

* Cambridge Philosophical Transactions, vol. vi.

sure, the maximum and minimum resultant tensions would be nearly equal, and consequently the directions of the fissures would be governed chiefly by accidental causes. Taking into consideration the causes which have been abstracted, it appears therefore that the directions of the fissures might in general have no necessary relations with that of the equator. If such a fissure commenced forming in the direction of any great circle, it would evidently continue to be propagated in the same general direction unless accidental causes should alter its course.

At some stages of the shell's existence the relation between the variable and the constant pressure might be such as to cause the maximum tension at any point to be in the direction of the tangent of the meridian, or parallel at that point, and hence, in this case at least, all simultaneously formed fissures should be parallel; the term parallel being used to mean that the tangents to the circles forming the prolongations of these fissures should all form equal angles at the points of intersection with the tangents of a great circle bisecting them.

VIII. ON THE EXISTENCE OF A ZONE OF LEAST DISTURBANCE IN THE SHELL.

18. During the process of the formation of the shell, the forces tending to fracture it, and those holding its parts together, will in general be continually varying. If the intensity of the former class of forces increased much more rapidly than that of the latter, there would be no limit to the disturbed part of the shell; if, on the contrary, the latter class of forces increased more rapidly in intensity than the former, some parts of the shell might after a certain time be comparatively undisturbed. It is even possible to conceive, if the constant pressure of the fluid against the shell be small, that in some parts of the shell the cohesive forces keeping its particles together might be always greater than the rupturing forces at these parts.

In order to form a general idea of the positions of such undisturbed portions of the shell, the variation of the effective pressure at any point in the shell must be considered. Of the two different ways in which the effective pressure could vary in going from the equator to the pole of the shell, it is evident, from Section IV., that the only one which it is necessary to consider is that of its decrease. In this case we may conclude, from article 2, that the pressure everywhere between the equator and parallel of mean pressure must be greater than that which exists at the same parallel, and that everywhere between this line and the pole the pressure will on the contrary be less. The disturbed portion of the shell must therefore be near the equator, and that line must divide it into two equal parts. In order that an undisturbed zone of the shell may exist, it is not necessary that the latter should be perfectly rigid; it may be capable of subsiding by the abstraction of a certain amount of the pressure of the fluid until its parts should rest in equilibrium. With a considerable variable pressure and a constant pressure insufficient to produce fractures in the shell, it is possible to conceive how this could occur, and hence the most general idea which can be formed of the undisturbed portion of the shell at either side of the equator, is that it must be a zone bounded by two parallels, both of which may become coincident with the equator, or evanescent, according as the pressures are very small or very great. If it be admitted that any great disturbances in the shell must have been chiefly due to the action of such pressures, it would be useful to examine the problem of the determination of the boundaries of the zone of least disturbance. By comparing the results of our investigation with observed phenomena, we may be able to deduce some conclusions directly applicable to geology.

I proceed therefore to determine the analytical expressions for the latitude of the limiting parallel of rupture on the side of the zone nearest to the equator. Let θ_2 represent this latitude, θ_3 the latitude of a parallel in the supposed undisturbed part of the shell.

It will appear that the general equation between the rupturing and cohesive forces upon which the position of the parallel of mean pressure depends, can, when the influence of temperature is abstracted, be made to contain but two independent variables θ_2 and a_1, or that it will be of the form $\mathrm{F}(\theta_2, a_1)=0$. The value required of θ_2 must evidently be a maximum, and hence we must determine it in general by elimination between the equations

$$\mathrm{F}(\theta_2, a_1),\ \frac{d\mathrm{F}}{da_1}+\frac{d\mathrm{F}}{d\theta_2}\cdot\frac{d\theta_2}{da_1}=0,\ \text{ or } \frac{d\mathrm{F}}{da_1}=0\,;$$

the fact of the resulting value being a maximum or minimum can be determined as usual by the sign of $\frac{d^2\theta}{da_1^2}$.

19. Let two infinite planes be conceived to pass through the axis of rotation of the shell, making so small an angle with each other that the curvature of the arc of the equator intercepted between them could be considered as insensible. The ratio of the rupturing to the cohesive forces in either of the opposite portions of the shell thus intercepted, will be the same as for the whole shell under our assumed conditions. The conditions for finding the section of rupture of one of these bands will therefore be the same as those for the entire shell.

Let $d\mathrm{L}$ represent an element of the area of the section of rupture, l its perpendicular distance from the neutral surface, l_1 and l_2 the perpendicular distances of the neutral surface from the outer and inner surfaces of the shell, s the resistance measured by the number of units of force required to rupture a square unit of section of the material of the shell at the distance l. The meaning generally attached, in works on mechanics, to the term neutral surface, is that here attached to it, namely, the surface at which the portions of the band subjected to the rupturing forces are neither compressed nor extended.

Let, as is generally assumed, s be a function of l, and let s_1 be its value at the distance l_1.

If the deflection of the band, in a state of tension and bordering on rupture, be very small, as must undoubtedly be the case from its physical structure, the directions of the rupturing forces will be perpendicular to the neutral surface, and therefore, from a well-known property of that surface, l_1 and l_2 will be the perpendicular distances of the centre of gravity of the section of rupture, from the outer and inner surfaces of the shell.

The force developed upon the element dL, opposed to the compression or extension of the material of the band at the distance l_1, will be $\frac{sl}{l_1}dL$, and its moment about the line formed by the intersection of the neutral surface with the section of rupture will be $\frac{s}{l_1}l^2dL$.

Let r_2 represent the radius drawn from the centre of the spheroid to the intersection of the neutral surface with the section of rupture; then the moment of the pressure P_1 between the section of rupture and the parallel of θ will be $r_2P\sin(\theta-\theta_2)dL'$, where θ stands for the latitude at any point of the band subjected to tension, and dL' an element of the area of the shell's inner surface. If we represent by b_1 the equatorial axis of the shell's inner surface, ω_1 the arc intercepted between the planes bounding the band, b_2 the equatorial axis of the neutral surface, and make $\varepsilon^2=\frac{b_1^2-a_1^2}{a_1^2}$, we shall have

$$dL=\frac{\omega_1 b_1\cos\theta}{1-\varepsilon^2\sin^2\theta},\quad r=b_2\sqrt{1-\varepsilon^2\sin^2\theta}.$$

As the sum of the moments of the pressures exerted on the band at either side of the section of rupture, when the shell is in the state bordering on rupture, must be equal to the sum of the moments of the cohesive forces at the same section, we shall have

$$\frac{1}{l_1}\int sl^2dL=\int_{\theta_2}^{\theta_3}r_2P_1\sin(\theta-\theta_2)dL=\omega b_1^2b_2(1-\varepsilon^2)\int_{\theta_2}^{\theta_3}\frac{P_1\sin(\theta-\theta_2)\cos\theta d\theta}{(1-\varepsilon^2\sin^2\theta)^{\frac{3}{2}}}.$$

But from article 3, $P_1=h(h_1+\cos^2\theta)$, making for brevity

$$h=(f-f_1)\int_0^{a'}\varrho a da,\quad h_1=\frac{\Pi}{h}-\frac{2}{3},$$

and remembering that here $\frac{\pi}{2}-\theta$ of that article is represented by θ. Hence

$$\frac{\int sl^2dL}{l_1\omega_1b_1^2b_2(1-\varepsilon^2)h}=\int_{\theta_2}^{\theta_3}\frac{(h_1+\cos^2\theta)\sin(\theta-\theta_2)\cos\theta d\theta}{(1-\varepsilon^2\sin^2\theta)\sqrt{1-\varepsilon^2\sin^2\theta}}$$

$$=h_1\cos\theta_2\int\cos\theta\sin\theta(1-\varepsilon^2\sin^2\theta)^{-\frac{3}{2}}d\theta-h_1\sin\theta_2\int\cos^2\theta(1-\varepsilon^2\sin^2\theta)^{-\frac{3}{2}}d\theta$$

$$+\cos\theta_2\int^{\theta}\cos^3\theta\sin\theta(1-\varepsilon^2\sin^2\theta)^{-\frac{3}{2}}d\theta-\sin\theta_2\int\cos^4\theta(1-\varepsilon^2\sin^2\theta)^{-\frac{3}{2}}d\theta.$$

The integrals multiplied by $\cos\theta_2$ can be easily found by ordinary methods, but the two multiplied by $\sin\theta_2$ cannot be given in a finite form. But from the nature of the problem one of the limits at least of the integrals must be an independent variable; hence the integrals multiplied by $\sin\theta_2$ can be determined only by developing the irrational factor in a converging series, and then integrating separately the resulting terms. For greater uniformity I develope in all the integrals the irrational factor, which becomes when developed

$$1+\frac{3}{2}\varepsilon^2\sin^2\theta+\frac{3}{2}\cdot\frac{5}{4}\varepsilon^4\sin^4\theta+\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\varepsilon^6\sin^6\theta+\text{ etc.}$$

Hence the sum of the four integrals will be, neglecting terms of the order ε^4,

$$h_1\left[\cos\theta_2\int\cos\theta\sin\theta d\theta-\sin\theta_2\int\cos^2\theta d\theta\right]+\cos\theta_2\int\cos^3\theta\sin\theta d\theta-\sin\theta_2\int\cos^4\theta d\theta$$

$$-\frac{3}{2}\varepsilon^2\{h_1[\cos\theta_2\int\sin^3\theta\cos\theta d\theta-\sin\theta_2\int\cos^2\theta\sin^2\theta d\theta]+\cos\theta_2\int\cos^3\theta\sin^3\theta d\theta-\sin\theta_2\int\cos^4\theta\sin^2\theta d\theta\},$$

On taking these integrals between the limits θ_2 and θ_3, the expression becomes

$$h_1\left[\frac{\cos\theta_2}{4}(\cos 2\theta_2-\cos 2\theta_3)-\frac{\sin\theta_2}{2}\left[\frac{1}{2}(\sin 2\theta_3-\sin 2\theta_2)+\theta_3-\theta_2\right]+\frac{\cos\theta_2}{8}\left[\frac{1}{4}(\cos 4\theta_2-\cos\theta_3+\cos 2\theta_2-\cos 2\theta_3\right]\right.$$

$$-\frac{\sin\theta_2}{4}\left[\frac{1}{3}(\sin 4\theta_3-\sin 4\theta_2)+\sin 2\theta_3-\sin 2\theta_2+\frac{3}{2}(\theta_2-\theta_3)\right]-\frac{3}{2}\varepsilon^2\left\{h_1\frac{\cos\theta_2}{8}\left[\frac{1}{4}(\cos 4\theta_3-\cos 4\theta_2)\right.\right.$$

$$\left.-(\cos 2\theta_3-\cos 2\theta_2)\right]+\frac{h_1\sin\theta_2}{8}\left[\frac{1}{4}(\sin 4\theta_3-\sin 4\theta_2)-\theta_3+\theta_2\right]+\frac{\cos\theta_2}{64}\left[\frac{1}{3}(\cos 6\theta_3-\cos 6\theta_2)\right.$$

$$\left.\left.-3(\cos 2\theta_3-\cos 2\theta_2)\right]+\frac{\sin\theta_2}{32}\left[\frac{1}{6}(\sin 6\theta_3-\sin 6\theta_2)+\frac{1}{2}(\sin 4\theta_3-\sin 4\theta_2)-\frac{1}{2}(\sin 2\theta_3-\sin 2\theta_2)-2(\theta_3-\theta_2)\right]\right\}$$

$$\left.\begin{array}{l}=[(a_1)\sin(\theta_3+\theta_2)\sin(\theta_3-\theta_2)+(a_2)\sin 2(\theta_3+\theta_2)\sin 2(\theta_3-\theta_2)+(a_3)\sin 3(\theta_3+\theta_2)\sin 3(\theta_3-\theta_2)]\cos\theta_2\\ -[(b_1)\cos(\theta_3+\theta_2)\sin(\theta_3-\theta_2)+(b_2)\cos 2(\theta_3+\theta_2)\sin 2(\theta_3-\theta_2)+(b_3)\cos 3(\theta_3+\theta_2)\sin 3(\theta_3-\theta_2)]\sin\theta_2\\ -(c_1)(\theta_3-\theta_2)\sin\theta_2.\end{array}\right\} \quad . \quad (37.)$$

Making for brevity

$$\left.\begin{array}{lll}(a_1)=\frac{1}{2}\left[h_1+\frac{1}{2}-\frac{3}{8}\varepsilon^2\left(h_1+\frac{3}{8}\right)\right], & a_2=\frac{1}{16}\left(1-\frac{3}{2}\varepsilon^2 h_1\right), & a_3=\frac{\varepsilon^2}{64},\\ (b_1)=\frac{1}{2}\left[h_1+1-\frac{3\varepsilon^2}{32}\right] & (b_2)=\frac{1}{16}\left[1+\frac{3}{2}\varepsilon^2\left(h_1+\frac{1}{2}\right)\right], & b_3=(a_3),\\ & (c_1)=\frac{1}{2}\left[h_1+\frac{3}{4}+\frac{3}{8}\varepsilon^2\left(h_1+\frac{1}{2}\right)\right]. & \end{array}\right\} \quad . \quad . \quad . \quad (38.)$$

20. I now proceed to determine the expressions for $\int sl^2 d\mathrm{L}$. In general the cohesive strength of an unit of surface of the section of rupture may be considered proportional to the number of molecules of which it is the section, and as the density ϱ_2 must at any point be also proportional to the number of molecules at that point, it follows that

$$s=s_1\frac{\varrho_2}{(\varrho_2)}.$$

The section of rupture may evidently be considered as a trapezoid, hence

$$d\mathrm{L}=\frac{\omega_1}{r_1}(r_2+l)\cos\theta_2 dl=\frac{\omega_1}{a_1}(1-\varepsilon^2\cos^2\theta)\cos\theta_2(a_1+l_2+l)dl=\frac{\omega_1}{a_1}(a_1+l_2+l)\cos\theta_2 dl$$

very nearly. Hence

$$\int sl^2 d\mathrm{L}=\frac{\omega_1 s_1\cos\theta_2}{a_1\varrho_2}\int_{-l_2}^{+l_1}\varrho_2(a_1+l_2+l)l^2 dl.$$

Let v represent the distance from the centre of the spheroid of any element of the section of rupture, v_1 the distance of the centre of gravity of that section, then

$$l_1=1-v_1,\ l_2=v_1-a_1,\ v_1=\frac{\int_{a_1}^{r}\varrho_2 v^2 dv}{\int_0^{r}\varrho_2 v dv}\ \dots\dots\dots\dots \quad (39.)$$

But $\varrho_2=\frac{C_2\sin a_1n_2}{a_1}$ (art. 9.) at the distance a_1; hence, substituting its value for the distance a_1+l_2+l, we obtain

$$\int\varrho_2(a_1+l_2+l)l^2dl=C_2\int l^2\sin(a_1+l_2+l)n_2dl.$$

Let $a_1+l_2+l=u$, and the preceding factor of C_2 becomes

$$\int u^2\sin un_2du-2(a_1+l_2)\int u\sin un_2du+(a_1+l_2)^2\int\sin un_2du.$$

Integrating between the limits l_1 and l_2, and we have

$$\int u^2\sin un_2du=\frac{1}{n_2^3}\{n_2^2a_1^2\cos a_1n_2-n_2^2(a_1+l_1+l_2)\cos(a_1+l_1+l_2)n_2$$

$$+2(a_1+l_2+l_2)n_2\sin(a_1+l_1+l_2)n_2-2a_1n_2\sin a_1n_2+2\cos(a_1+l_1+l_2)n_2-2\cos a_1n_2\},$$

$$\int u\sin un_2du=\frac{1}{n_2^2}[(a_1+l_1+l_2)n_2\cos(a_1+l_1+l_2)n_2+a_1n_2\cos a_1n_2+\sin(a_1+l_1+l_2)n_2-\sin a_1n_2],$$

$$\int\sin un_2du=\frac{1}{n_2}[\cos a_1n_2-\cos(a_1+l_1+l_2)n_2].$$

But $l_1+l_2=1-a_1$; hence

$$\int_{-l_2}^{+l_1}\varrho_2(a_1+l_2+l)l^2dl=C_2\left\{\frac{1}{n_2^3}[(n_2^2a_1^2-2)\cos a_1n_2-(n_2^2-2)\cos n_2+2n_2(\sin n_2-a_1\sin a_1n_2)]\right. \quad (40.)$$

$$\left.+\frac{2(a_1+l_2)}{n_2^2}[\sin a_1n_2-a_1n_2\cos a_1n_2-\sin n_2+n_2\cos n_2]+\frac{(a_1+l_2)^2}{n_2}(\cos a_1n_2-\cos n_2)\right\}.$$

To determine l_1 and l_2, I substitute for ϱ_2 its value in (39.) and integrate, then

$$v_1=\frac{n_2(a_1\cos n_2a_1-\cos n_2)+\sin n_2-\sin a_1n_2}{n_2(\cos a_1n_2-\cos n_2)}$$

$$l_1=\frac{(1-a_1)\cos a_1n_2}{2\sin\frac{1}{2}(1+a_1)n_2.\sin\frac{1}{2}(1-a_1)n_2}-\frac{1}{n_2}\cot\frac{1}{2}(1+a_1)n_2$$

$$l_2=\frac{1}{n_2}\cot\frac{1}{2}(1+a_1)n_2-\frac{(1-a_1)\cos n_2}{2\sin\frac{1}{2}(1+a_1)n_2.\sin\frac{1}{2}(1-a_1)n_2}.$$

If we make

$$j=\left\{(1-a_1)\cot\frac{1}{2}(1-a_1)n_2-1\right\}\cot\frac{1}{2}(1+a_1)n_2,$$

we shall have

$$l_1=\frac{1-a_1+j_2}{2},\ l_2=\frac{1-a_1-j_2}{2}.\ \dots\dots\dots\dots \quad (41.)$$

It appears, from the value of n_2 deduced by observation, that when $1-a_1$ is a small fraction, j will be small compared with l_1 or l_2; hence in general, when the shell is thin,

$$l_1=l_2=\frac{1}{2}(1-a)_1.$$

21. The function in which θ_2 is to be made a minimum is by art. 19,

$$(A)-[(a_1)\sin(\theta_3+\theta_2)\sin(\theta_3-\theta_2)+(a_2)\sin 2(\theta_3+\theta_2)\sin 2(\theta_3-\theta_2)+(a_3)\sin 3(\theta_3+\theta_2)\sin 3(\theta_3-\theta_2)]$$
$$+[(b_1)\cos(\theta_3+\theta_2)\sin(\theta_3-\theta_2)+(b_2)\cos 2(\theta_3+\theta_2)\sin 2(\theta_3-\theta_2)+(b_3)\cos 3(\theta_3+\theta_2)\sin 3(\theta_3-\theta_2)]\tan\theta_2$$
$$+(c_1)(\theta_3-\theta_2)\tan\theta_2=0 \text{ [(A) being a function of } a_1],$$

or

$$(A)-[(a_1)\sin(\theta_3+\theta_2)-(b_1)\cos(\theta_3+\theta_2)\tan\theta_2]\sin(\theta_3-\theta_2)-[(a_2)\sin 2(\theta_3+\theta_2)-(b_2)\cos 2(\theta_3+\theta_2)\tan\theta_2]\sin 2(\theta_3-\theta_2)$$
$$-(a_3)[\sin 3(\theta_3+\theta_2)-\cos 3(\theta_3+\theta_2)\tan\theta_2]+(c_1)(\theta_3-\theta_2)\tan\theta_2=0. \quad \ldots \ldots \quad (42.)$$

From this expression we can deduce the following conclusions:—

1st. If θ_2 be less than $\frac{\pi}{2}$, θ_3 cannot be zero unless $(A)=0$.

2nd. If $\theta_2=\frac{\pi}{2}=\theta_3$, $\cos(\theta_3+\theta_2)=-1$, $\cos 2(\theta_3+\theta_2)=0$, $\cos 3(\theta_3+\theta_2)=-1$,

and consequently

$$(A)=(b_1)+(b_3)-(c_1)=\frac{1}{8}\left[1-\frac{3}{2}\varepsilon^2\left(\frac{1}{2}+h_1\right)\right]$$
$$h_1=\frac{2}{\varepsilon^2}(1-8(A))-\frac{1}{2}.$$

3rd. In order that in this case $\Pi=h$, we must have $(A)=\frac{1}{8}\left(1-\frac{5}{12}\varepsilon^2\right)$.

From article 20,

$$(A)=\frac{4s_1\int_0^{\frac{1}{2}(1-a_1)}\varrho_2(1+a_1+2l)l^2dl}{(1-2e_1)(1+e_1)^2(1-a_1^2)a_1^2h}.$$

The value obtained for the numerator of this expression, shows that for no conceivable value of a_1 (A) can vanish; and therefore from the first of the preceding conclusions, it may be deduced that if the shell be fixed under any parallel, it must be also fixed under some other parallel at the same side of the equator and at a distance from it, depending on the cohesive strength of the material of the shell. Consequently in the case of $\theta_2<\frac{\pi}{2}$ a zone of least disturbance should exist. In general it is evident that the value of (A) $\left(\text{supposing } \theta_3=\frac{\pi}{2}\right)$, which would make θ_2 a maximum, must be extremely small; consequently, from the second and third conclusions, if $\theta_2=\frac{\pi}{2}$, or if no zone of least disturbance exist, Π must be considerably greater than h.

IX. CALCULATION OF SOME OF THE CONSTANTS CONTAINED IN THE FORMULÆ OF THE PRECEDING SECTIONS.

22. 1st. To find the numerical value of n_2 and n. It is evident that when D represents the mean density of the earth, we should have

$$\frac{1}{n_2^2}(1-n_2\cot n_2)=\frac{\mathrm{D}}{3(\varrho_2)},$$

(ϱ_2) being the density of the shell's outer surface, or in other words, the density of the first stratum solidified from the primitive fluid mass constituting the earth. To find D, I take a mean of the results of the best experiments which have been made for its determination, namely, of the experiments of CAVENDISH, BAILY and REICH, or respectively of the numbers 5·48, 5·68 and 5·44. The mean of these results is 5·53 nearly. The density of granite, the crystalline rock which seems to form the base of all the sedimentary formations, is evidently that which must be used for $(\varrho_2.)$ This remark is important, because in all comparisons heretofore made of the mean and surface density of the earth, the mean density of the sedimentary rocks has been erroneously taken into consideration. From a comparison of the densities of granite obtained in different countries on the authority of different geological and engineering works, I have decided that the mean density of the rock cannot be less than 2·7. Adopting this value, the above equation becomes

$$\frac{1}{n_2^2}(1-n_2\cot n_2)=\cdot 682716,$$

which is approximately satisfied by making

$$n_2=\frac{14227}{18000}\pi=142^\circ\ 16'\ 22''.$$

To find n, we have

$$\frac{n^2}{1-n\cot n}=\frac{(k)n_2^2}{1-n_2\cot n_2}=\frac{(k)}{\cdot 682716}.$$

Let $(k)=\cdot 7481$, its least value found by experiment. Then

$$\log\left(\frac{n^2}{1-n\cot n}\right)=\cdot 0397196.$$

By trials I find that when $n=154^\circ\ 25'$, the quantity at the left side of the above expression is ·0396799, and when $n=154^\circ\ 24'$, it becomes ·0398333; hence it lies between both of these values.

When $n=154^\circ\ 24'\ 30''$,

$$\log\left(\frac{n^2}{1-n\cot n}\right)=\cdot 0397196,$$

hence this value will serve for a first approximation.

When $(k)=\cdot 896$, the greatest value found by experiment, we should have

$$\log\left(\frac{n^2}{1-n\cot n}\right)=\cdot 1180679.$$

The value $n=147° 30'$ would make this ·1188293, hence this value of n would be a little too small.

2nd. To find p' and λ. From the fourth chapter of the fourth book of the work of M DE PONTÉCOULANT, it will be perceived that

$$p=\frac{4ln}{3m^2(1+\gamma)\cos h},$$

where l represents the mean movement of the equinoxes at the time when h represents the apparent obliquity of the ecliptic, and γ the ratio of the moon's action on the earth compared to that of the sun, n and m being constants, the former depending on the earth's rotation, and the latter on the sun's mean movement. If the variation of h be referred to the plane of the ecliptic in 1800,

$$h=23° 27' 55'', \ l=50''\cdot363541,$$

and also

$$n=360°\cdot98561, \ m=359° 59'\cdot37;$$

and when m is referred to the same unit of time as n,

$$m=0°\cdot98561.$$

Of the three different methods by which γ can be determined, I select that depending on the phenomena of nutation. I do so because it seems that astronomers have taken much pains to determine the numerical coefficient which depends on these phenomena, and on which the value of γ depends. If we represent the coefficient of nutation by N*, we shall have

$$\frac{\gamma}{1+\gamma}=\frac{N}{13''\cdot36926}, \text{ or } \gamma=\frac{N}{13''\cdot36926-N}.$$

The following values have been deduced for N

By ROBINSON . . .	9·234″
By BRINKLEY . . .	9·25
By LINDENAU . . .	8·977
By PLANA	8·925

the mean of which is 9″·0965, and therefore $\gamma=2\cdot128951$;

$$\begin{aligned} \log(1+\gamma)&=\ \cdot4953988 \\ \log m&=6\cdot1125773 \\ \log\cos h&=9\cdot9625122 \\ \log 3&=\ \cdot4771213 \\ \hline &\ 7\cdot0476296 \end{aligned} \qquad \begin{aligned} \log l&=1\cdot7021163 \\ \log\frac{n}{m}&=2\cdot5637896 \\ \log 4&=\ \cdot6020600 \\ \hline &\ 4\cdot8679659 \\ &\ 7\cdot0476296 \\ \hline \log p'&=\bar{3}\cdot8203363 \end{aligned}$$

* PONTÉCOULANT, Théorie, &c., tome iv., Note 3, p. 654.

If we represent by λ' the coefficient of the variation of gravity at the earth's surface, found by pendulum experiments, and by λ'' the same coefficient deduced from observation of the moon's inequalities in longitude and latitude, then

$$\lambda'=\frac{5}{2}m-e', \quad \lambda''=\frac{5}{2}m-e'',$$

e' and e'' being the corresponding ellipticities of the earth's surface found by the ordinary theory. Let, in accordance with the latest calculations,

$$e'=\frac{1}{288}, \quad e''=\frac{1*}{300},$$

then λ, the most probable value of the coefficient of the variation of gravity, becomes

$$\frac{5}{2}m-\frac{1}{293\cdot88}.$$

3rd. To find the numerical value of Q in Section V. (equation 36). From the value of n_2 just obtained,

$$1-\frac{6}{n_2^2}=\cdot026872, \quad 2+\left(1-\frac{6}{n_2^2}\right)q_2=2\cdot11311.$$

If we make, in accordance with observation,

$$e=\frac{1}{300}, \quad m=\frac{1}{289},$$

and use the value of λ above obtained,

$$\frac{2}{3}\frac{2e+m-\lambda}{p'}q_2=\frac{8122.q_2}{5\times578\times2880p}=2\cdot07068,$$

$$Q=\cdot045253\times\sin n_2=\cdot0260259.$$

GEOLOGICAL DEDUCTIONS FROM THE FOREGOING INVESTIGATIONS.

(1.) *The Stability of the Axis of Rotation of the Earth.*

23. The conclusion arrived at in article 14, shows that if the rotation of the earth were originally stable about its axis, it would continue to rotate in the same way for ever.

The action of exterior bodies has been heretofore alone examined in considering the question of the position of the earth's axis of rotation within it. From this examination, it results that the action of such bodies would be incapable of producing any change in the position of the axis, and hence, if such a change were at all possible, it should be produced by some interior action by which the distribution of the particles composing the earth would be changed. It is admitted, that if the earth were fluid and in rotation with an angular velocity differing but little from its present angular velocity, it would rotate stably about its shorter axis. During the process of its successive solidification, it might happen that the new arrangements of the particles

* See HUMBOLDT'S Kosmos, Bd. I. s. 174, and PONTÉCOULANT, tome iv. p. 486.

might be such as to disturb the rotation, not only by increasing or lessening the angular velocity, but also by changing the position of the axis. It appears however, from the article cited, that the difference of the greatest and least moment of inertia of the earth must progressively increase during the process of solidification, and hence that the stability of rotation must continually increase until it reaches its limit when the mass shall have arrived at the entirely solid state.

Thus not only does a question closely connected with geological theory seem to be definitively settled, but also the future stability of the earth's rotation appears to be completely assured.

(2.) *The Thickness of the Earth's Solid Crust.*

24. In Section II. expressions have been obtained in which the variation of gravity at the earth's surface is a function of the radius and ellipticity of the fluid nucleus supposed to exist within it, from which it will be possible to deduce the limiting values of that radius, and consequently of the thickness of the solid shell. If we refer to the general expression (20.), it will be perceived that the greater is e_1, the less the thickness of the shell; hence we would be able to obtain its greatest thickness consistent with observation, other quantities remaining unchanged, by giving to e_1 its least value. But from Section IV. the least value of e_1 is e, or the ellipticity of the surface of the shell, hence in this case $m_1=m$, and

$$\lambda=\left[\frac{5}{2}-3(f(a)-\mu_1 a_1^2 f(a_1)\right]m+2e-3\mu a_1^2 e.$$

The greatest value which a_1 can receive will depend on the limits imposed on the values of the functions depending on the earth's internal constitution; and if we give to these values favourable to a large value of a_1, e_1 must very nearly be equal to e. The above expression for λ will therefore suffice for this case by attributing the proper values to the functions $f(a)$ and $f(a_1)$. But when $e_1=e$, we have, by the expression deduced in article (5.),

$$\lambda=2e+m+\frac{3}{2}\left(\frac{5a_1^5\nu\sigma_1 m}{5\sigma_1-3}-2a_1^5\nu e-\frac{3m}{5\sigma-3}\right);$$

ν representing the same quantity as in article (13.), σ_1 will differ but little from σ when $e_1=e$, and with a continuous law of density of the strata of the shell; hence in a first approximation we may assume their equality. But $\lambda=\frac{5}{2}m-(e)$, (e) being the value of e, deduced by the ordinary theory from observation of the variation of gravity at the earth's surface, hence

$$a_1=\nu^{-\frac{1}{5}}\left[1-\frac{2}{3}\,\frac{(5\sigma-3)(e)-eI}{6e-5\sigma(2e-m)}\right]^{\frac{1}{5}}.$$

To find the limiting values of a_1, we must make this a maximum or a minimum. If we suppose ν to differ but little from 1, it will be nearly constant, and therefore

$$\frac{da_1}{d\sigma}=\frac{2}{3}\left[1-\frac{2}{3}\,\frac{p\sigma-p'}{q'-q\sigma}\right]^{-\frac{4}{5}}\left[\frac{qp'-pq'}{(q-\sigma q)^2}\right],$$

$$\frac{d^2a_1}{d\sigma^2}=\frac{16}{15}\left\{\frac{1}{q'-\sigma q}-\frac{pq'-qp'}{3\left(1-\frac{2}{3}\frac{p\sigma-p}{q'-q\sigma}\right)}\right\}\left(1-\frac{2}{3}\frac{p\sigma-p'}{q'-q\sigma}\right)^{-\frac{4}{5}}\frac{pq'-p'q}{(q'-\sigma q)^2};$$

putting $p=5[(e)-e]$, $p'=3[(e)-e]$, $q'=6e$, $q=5(2e-m)$.

But $1-\frac{2}{3}\frac{p\sigma-p'}{q-q\sigma}$ is always less than 1, and $q'-\sigma q$, $pq'-p'q$ are both positive and small quantities. Hence $\frac{d^2a_1}{d\sigma^2}$ is positive, and therefore the value of σ, deduced by making $\frac{da_1}{d\sigma}=0$, gives a minimum for a_1. This value is evidently $\sigma=\frac{3q'+2p'}{3q+2p}$.

On substituting this in the value of a_1, we shall have $a_1=0$: $\frac{da_1}{d\sigma}=0$ would be also satisfied by making $\sigma=\infty$, but the supposition that ν is finite and little different from 1, excludes this value. Hence a_1 must increase from its least value as σ decreases to its least value, but this is when $\sigma=1$, or when the shell is supposed to be homogeneous; consequently the greatest value of a_1 will be determined from the equation

$$a_1^5=\frac{\lambda+\frac{5}{4}m-2e}{3\left(\frac{5}{4}m-e\right)}=\frac{2}{3}+\frac{\lambda-\frac{5}{4}m}{3\left(\frac{5}{4}m-e\right)}=\frac{2}{3}+\frac{1}{3}\,\frac{\frac{5}{4}m-(e)}{\frac{5}{4}m-e}.$$

From Section IX. $(e)=\frac{1}{294}$ nearly; and using for e and m the values respectively $\frac{1}{300}$ and $\frac{1}{289}$, we shall obtain

$$a_1^5=\frac{1}{3}\left[2+\frac{157\times 50}{172\times 49}\right]=\cdot 97714;$$

hence $a_1=\cdot 99539$, $1-a_1=\cdot 00461$, and therefore, consistently with observation, the least thickness of the earth's crust cannot be less than 18 miles.

From Section III. it appears that as (k) diminishes the more will the constitution of the nucleus and shell be different, and consequently the least value which can be attributed to (k) from observation, may be assumed to give the least value of a_1 in equation (35.). By using this equation together with (29.), I find, after a few trials, that with $a_1=\cdot 85$, the approximate value of Q would be $\cdot 023441$. But Q increases with a_1, hence the greatest thickness of the shell cannot, on the above assumption, be greater than $\cdot 15$, or 600 miles nearly.

(3.) *The Earth's Ellipticity when entirely fluid.*

25. By the aid of equation (11.), Part I., the following values of the earth's primitive ellipticity have been obtained with different values of (k).

With $(k)=\cdot 7481$, $E=\frac{1}{302\cdot 22}$,

$(k)=\cdot 896$, $E=\frac{1}{310\cdot 74}$,

$(k)=1\cdot$, $E=\frac{1}{316}$.

The value of the earth's moment of inertia used in these calculations, was deduced by using the value of p found in Section IX., and that of λ found from the lunar inequalities. If the value of the latter quantity, found in the section referred to, were used, the resulting values of E would all be a little smaller; it may therefore be concluded, that *the earth's primitive ellipticity was less than its present ellipticity, although the difference between them may be neglected.*

(4.) *The Direction of great Lines of Elevation on the Surface of the Earth.*

26. If a zone of least disturbance existed near the parallel of mean pressure, the directions of great lines of elevation should be nearly parallel or perpendicular to the equator. Its non-existence there, which observation seems to show, proves at least that the variable pressure did not predominate over the constant pressure.

As yet, observation seems to prove that such a zone does not exist on the earth's surface; and hence from Section VIII. we must provisionally conclude that the constant pressure greatly predominated over the variable pressure, and consequently that the directions of lines of elevation must be comparatively arbitrary. Geological and geographical observations present results which are generally in accordance with these views.

(5.) *The Existence of great Friction and Pressure at the Surface of Contact of the Nucleus and Shell.*

27. The conclusions arrived at in Section IV., combined with the important result obtained by Mr. Hopkins in his second memoir on Physical Geology*, show that great friction and pressure must exist between the shell and fluid nucleus. The result alluded to, as quoted in Mr. Hopkins's third memoir, is

$$\text{“}\, P' - P_1 = \left(1 - \frac{\varepsilon}{\varepsilon_1}\right)\left(1 - \frac{\eta}{1 + \frac{h}{q^5 - 1}}\right) P_1,$$

where P denotes the precession of a solid homogeneous spheroid of which the ellipticity $= \varepsilon_1$, that of the earth's exterior surface, and P' the precession of the earth, supposed to consist of a heterogeneous fluid nucleus contained in a heterogeneous spheroidal shell, of which the interior and exterior ellipticities are respectively ε and ε_1, the transition being immediate from the entire solidity of the shell to the perfect fluidity of the mass.” This result was found on the hypothesis of the non-existence of friction and pressure from molecular causes at the surface of contact of the shell and nucleus; if this hypothesis were true, we should have in general, from articles 12 and 13, $P' \angle P_1$, or at least, we could not have $P' \,7\, P_1$. This result is so different from that obtained by this observation, that we are entitled to assume that the motion of rotation of both shell and nucleus takes place nearly as if the mass were entirely solid. It must evidently result, that at the surface of contact of the solid and fluid, a con-

* Philosophical Transactions, 1840, p. 207.

siderable amount of friction and pressure should necessarily exist. From physical reasons, there is also a great probability of the truth of this proposition. The highly crystalline structure which must necessarily be assigned to the shell's inner surface, combined with the viscidity of the strata of the nucleus in immediate contact with it, would evidently, if sufficiently great, tend constantly to equalize the motions of both shell and nucleus, and to cause the whole to rotate as one mass. Of the truth of the fundamental assumption in article 1, I may therefore venture to hope no reasonable doubt can exist.

(6.) *The Amount of Elastic Gases given off at the Surface of the Nucleus during different Geological Epochs.*

28. If, in the strata of the nucleus, elastic gases were confined by the compression of the superior strata, it must follow from Section III. that during the solidification of the upper parts of the fluid the rest will tend to expand, and consequently to set free the confined gases. If the amount of elastic gases in a cubic unit of any stratum be proportional to the pressure to which that stratum is subjected, it must follow that the quantity evolved on the solidification of the surface stratum must be proportional to the amount of contraction of that stratum in solidifying. The expansion of the other strata, yet in a fluid state, would also tend to produce an evolution of such gases. The evolution of gases from the solidifying stratum would evidently be proportional to $(1-k)a_1^2$, and from any other stratum with the radius a_1, proportional to the density ϱ of that stratum. But it has been shown in Section III. that in general ϱ and $(1-k)$ decrease as a_1 decreases; hence we may conclude that the amount of elastic gases given off from the surface of the nucleus rapidly decreases as the thickness of the shell increases.

(7.) *The Distribution of the Waters on the Surface of the Globe.*

29. The expressions for the variation of gravity obtained in Section II., show that if the angular velocity of rotation of the earth remained unchanged, the waters on its surface would tend to accumulate towards the equator, by the gradual thickening of the shell and the consequent change in the direction of gravity.